General Relativity in the Solar System

Thesis[1]

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Hungarian original: April 1st, 2008
English translation and update: November 1st, 2008

Abstract:

The goal of this paper is to review observable phenomena in the Solar System where gravitation is playing a role, from the viewpoint of general relativity. After some mathematical basics, the relevant vacuum solutions of the Einstein field equations, and the resulting Schwarzschild and Kerr[2] spacetimes are derived and examined. They are treated as self-sufficient models for gravitation, and the sole tools necessary for describing not only relativistic, but all major phenomena in celestial mechanics. Hypothetical, not observable or not relevant consequences of the theory for the Solar System were omitted from the discussion.

1 This is a translated and updated version of the author's MSc thesis at the University of Pécs in 2008
2 There is much emphasis in this paper on the demonstration of gravitation with purely general relativistic tools, thus post-Newtonian approximations are avoided

Contents

Foreword

General relativity provides the most complete description of gravitational interaction, as we know it today. The validity of the equations published by Albert Einstein in 1915[2] was proved by numerous experiments and observations during the last 93 years. Although at the frontiers of this physical model many oddities and paradoxes have arisen, there is no known gravitational phenomenon today that cannot be described within the framework of the theory[3]. This paper is not pushing the boundaries, merely examining the more or less ordinary phenomena occurring in the Solar System.

In classical or deterministic physics, the most important difference between gravitation and other interactions is, that it is not described by a real force, the motion of celestial bodies and falling objects has purely geometrical reasons. Ironically, gravitation was the original inspiration to create the concept of force, which went on to determine mechanics. What makes it special is that what controls both the fall of an apple and the orbit of Moon is also determining space and time, where those objects exist, thus creating the framework for every other interaction we know of.

The philosophical thoughts and ideas, from Galileo Galilei (relativity principle) through Rudolf Clausius (direction of processes) to James Clerk Maxwell (constant speed of light), that brought to the world and then expanded Newton's mechanics, materialize in a numerically tangible form of Einstein's relativity theories (who also contributed to our world view with the recognition of the equivalence principle). This fact can't be overemphasized, because although the reader is surely familiar with the work of the rightly popular scientist, many people regard relativity as one of the most difficult intellectual creation of all. However this undoubtedly brilliant revelation was an organic part of the development of our knowledge on the world, what doesn't mean that it would be alien or incomprehensible at all.

Therefore it is clear, that the true understanding of gravitation can be attained only by the study of general relativity. The Newtonian force-formula is not simply a borderline case of the broader theory. Several gravitational phenomena exists, that are hard to described by tools based on the classical concept of force, for example geodesic precession, or frame dragging. Therefore when gravitation is mentioned in the educational system, it cannot happen without also talking about general relativity, in the appropriate way for different age groups of course. It is the aim of this paper to bring closer the theory to the reader by examining and explaining gravitational phenomena in our solar neighbourhood, sometimes with original methods, including those, that are usually explained using Newtonian theory.

3 since even frame-dragging and – indirectly – gravitational waves have got observational evidence, calling it a theory has only historical reasons by now

Introduction

Maybe our senses mislead us, but we don't notice that while we sit calm in a room, Earth is actually speeding at nearly $30\frac{km}{s}$ around the Sun. Most of the time, we can't even tell if we are standing still or moving, inside of an ocean liner[1]. The fact is, that not only us, but (in the ideal case) also neither our instruments can tell the difference. They may be inaccurate, but it is also possible that there is a principal reason keeping us from telling our absolute velocity.

Our journey in time in one direction from the past to the future, and the ordering between cause and effect, is such a self-evident experience, that it is surprising that we have to explicitly state it as a condition, since serious logical flaws and paradoxes would arise, if it were not so. But from a practical point of view, we can't tell anything except that we have not yet observed the contrary. We also have to take into account, that nothing has been said on the simultaneity of events.

Light is travelling so fast, for a long time in history nobody ever got the idea, that it may have a finite velocity. We recognize it only at astronomical distances, or with the help of our instruments, having much better reaction time, than our naked eyes. It is an important fact that its speed in vacuum is always the same and constant, for all observers, regardless of their motion. We trust this observation so much, that we use it for the base of the definition of meter in SI units. If it were not so, a very fast observer could outrun light, and measure a different value for its velocity, thus measure his/her absolute speed, something impossible, as we believe.

Astronauts in an aeroplane on a paraboloidal path[4] can experience weightlessness for a short period of time. Seats of the visitors in fun park simulators are tipped back to simulate gravity, they feel only their own weights, but they are led to believe that they accelerate. If the simulator would actually move away from its position, not only rotate, the person sitting inside could not tell the difference. There are two indistinguishable phenomena again, let us declare that they are the same.

Formally the above principles are:

1. equivalence of uniformly moving observers
   $K \equiv K'$ — *Galilei, equivalence of inertial frames*
2. not reversible order of casual sequences
   $t_1 \leqslant t_2 \leftarrow\rightarrow t'_1 \leqslant t'_2$ — *Clausius, principle of entropy*
3. constant speed of light
   $c = 299{,}792{,}458\frac{m}{s}$ — *Maxwell, electromagnetic waves*
4. equivalence of free moving and falling observers
   $K \equiv G$ — *Einstein, principle of equivalence*

The above principles are direct descendants of the ideas from renaissance and the age of enlightenment, thus general relativity is one of the great classic physical models of nature.

Index notation with Einstein's summation convention is used, with the following rules:

dimensions	*free indices*	*summation indices*
3D (1, 2, 3) and general (1 … n)	i, j, k, l, m, n	a, b, c, d, e, f
4D (0, 1, 2, 3)	η, κ, μ, ν, ξ, σ	α, β, γ, δ, ε, ζ

4 The famous „Vomit Comet” of the NASA astronaut training program

Derivations by affine parameters are noted by dots above the quantity. Partial first and second derivatives are noted with the short form:

$$\frac{\partial}{\partial x^i}=\partial_i \qquad \frac{\partial}{\partial x^i\cdot\partial x^j}=\partial^2_{ij}$$

1. Mathematical basics

If physics is the story, then mathematics is the language that is used to tell it. The words are vectors and tensors, and the grammar is called Riemannian geometry. General relativity is essentially applied differential geometry, it is praised for the elegant application of mathematics in the real world. Physical quantities are directly interpreted into geometrical properties of manifolds. The spacetime is associated to a four-dimensional surface, and every properties of gravitation are actual mathematical properties of that surface. Let us recall the basic rules without proving them, so we have something to work with when we go on to the actual derivations of this paper.

Measurement in spacetime begins with distance. We have to declare some rules of distances, so called axioms. Let us call $d()$ the distance function, that accepts two points as parameters, and the rules are:

distance of a point *a* from itself equals zero:

$$d(a,a)=0 \tag{1.1}$$

distance of point *a* from point *b* is equal to the distance of point *b* from point *a*:

$$d(a,b)=d(b,a) \tag{1.2}$$

We allow the metric to have negative distances, or even zero distances between distinct points. It is important to note, that the triangulation rule doesn't apply in this case.

Let us set up a two biorthogonal bases, they are marked with upper and lover indices, and the vectors in those bases, called covariant and contravariant vectors respectively:

$$\vec{e}_i\cdot\vec{e}^j=\delta_i^j \qquad \vec{v}_i=\vec{e}^a\cdot v_a \qquad \vec{v}^i=\vec{e}_a\cdot v^a \tag{1.3}$$

We create the metric tensor's and its reciprocal components by scalar multiplying the base vectors:

$$\vec{e}_i\cdot\vec{e}_j=g_{ij} \qquad \vec{e}^i\cdot\vec{e}^j=g^{ij} \tag{1.4}$$

The metric tensor is used to raise and lower indices of vectors:

$$v_i=g_{ia}\cdot v^a \qquad v^i=g^{ia}\cdot v_a \tag{1.5}$$

And we define the square of infinitesimal invariant distance on the manifold, that complies to the distance rules above:

$$ds^2 = g_{ab} \cdot dx^a \cdot dx^b \tag{1.6}$$

Local straight lines in the curved surface are described by the geodesic equations:

$$\ddot{x}^i + \Gamma^i_{ab} \cdot \dot{x}^a \cdot \dot{x}^b = 0 \tag{1.7}$$

Where *x* denotes coordinates, points above quantities are derivatives by the affine parameter, and the metric connection can be derived from the metric tensor and its first partial derivatives:

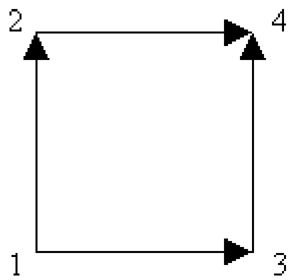
$$\Gamma^i_{jk} = \frac{1}{2} \cdot g^{ia} \cdot (\partial_k g_{ia} + \partial_i g_{ak} - \partial_a g_{ki}) \tag{1.8}$$

Trajectories in general relativity depend only on the initial conditions of coordinates and their first derivatives, just like in any other classical model, despite the fact, that time is one of the variable coordinates. Therefore in practice the only condition a quantity must satisfy to become a suitable affine parameter is to change monotonic during the movement. This can be the proper time of the moving body, or in many cases, even one of the coordinates, for example the time axis in the the global coordinate system.

Parallel displacement along a geodesic curve with the tangent vector *u* is described by the following equation:

$$_2u^i = {}_1u^i - \int \Gamma^i_{ab} \cdot u^a \cdot dx^b \tag{1.9}$$

The only quantity that fully describes the geometry of a surface internally is the curvature tensor, it can be derived using infinitesimal parallel displacements of vectors along the sides of a parallelogram:



(1 → 2 → 4) or (1 → 3 → 4)

We perform the displacements along both paths, and calculate their difference, that depends only on the enclosed surface. The result can be expressed only by connections and its first derivatives:

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{ak} \cdot \Gamma^a_{jl} - \Gamma^i_{al} \cdot \Gamma^a_{jk} \tag{1.10}$$

We limit our investigation to astronomical phenomena in the Solar System, where Einstein's equations for empty spacetime apply, it can be derived with the action principle:

$$R_{ij} - \frac{1}{2} \cdot g_{ij} \cdot R = 0 \tag{1.11}$$

Where the Ricci-tensor is the first contraction of the curvature tensor:

$$R_{jl} = R^a_{jal} = \partial_a \Gamma^a_{jl} - \partial_l \Gamma^a_{ja} + \Gamma^a_{ba} \cdot \Gamma^b_{jl} - \Gamma^a_{bl} \cdot \Gamma^b_{ja} \quad (1.12)$$

All properties of space and time are consequences of geometry, including phenomena in special relativity. Mathematical quantities described above have direct physical meanings, for example first derivatives of contravariant vectors are velocities, first derivatives of covariant vectors are momenta, the connection represents most fictitious forces (the Coriolis force and the force of gravity are among them) and the curvature tensor represents all components of the tidal force.

2. Spherically symmetric coordinate system

We set up a central-symmetric coordinate system in flat spacetime, and calculate the geodesic equations from the invariant distance. This example demonstrates the path we will take in the rest of the paper, and we will use the resulting equations when we compare the Schwarzschild solution to the Newtonian limit.

Start with a spherical coordinate system, that we extend with radial and timelike coordinates. The invariant distance:

$$ds^2 = c^2 \cdot dt^2 - dr^2 - r^2 \cdot d\vartheta^2 - r^2 \cdot \sin^2(\vartheta) \cdot d\varphi^2 \quad (2.1)$$

We use it to identify the components of the covariant and contravariant metric tensors, and to calculate their partial derivatives:

$$g_{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \cdot \sin^2(\vartheta) \end{pmatrix} \qquad g^{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \cdot \sin^2(\vartheta)} \end{pmatrix}$$

$$\partial_1 g_{22} = -2 \cdot r \qquad \partial_1 g_{33} = -2 \cdot r \cdot \sin^2(\vartheta) \qquad \partial_2 g_{33} = -2 \cdot r \cdot \cos(\vartheta) \cdot \sin(\vartheta) \quad (2.2)$$

The next step is to identify the nonzero connection coefficients:

$$\Gamma^\eta_{\kappa\mu} = \frac{1}{2} \cdot g^{\eta\alpha} \cdot (\partial_\mu g_{\eta\alpha} + \partial_\eta g_{\alpha\mu} - \partial_\alpha g_{\mu\eta})$$

$$\Gamma^1_{22} = -r \qquad \Gamma^1_{33} = -r \cdot \sin^2(\vartheta) \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{1}{r}$$

$$\Gamma^2_{33} = -\cos(\vartheta) \cdot \sin(\vartheta) \qquad \Gamma^3_{13} = \Gamma^3_{13} = \frac{1}{r} \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot(\vartheta) \quad (2.3)$$

We don't calculate the partial derivatives of the connection coefficients because they are only needed for the curvature tensor, and that is zero in the case of flat spacetime. The geodesic equations are:

$$(0) \qquad c\cdot\ddot{t}+\Gamma^{0}_{\alpha\beta}\cdot\dot{x}^{\alpha}\cdot\dot{x}^{\beta}=\ddot{t}=0 \tag{2.4}$$

$$(1) \qquad \ddot{r}+\Gamma^{1}_{\alpha\beta}\cdot\dot{x}^{\alpha}\cdot\dot{x}^{\beta}=\ddot{r}+\Gamma^{1}_{22}\cdot\dot{\vartheta}^{2}+\Gamma^{1}_{33}\cdot\dot{\varphi}^{2}=\ddot{r}-r\cdot\dot{\vartheta}^{2}-r\cdot\sin^{2}(\vartheta)\cdot\dot{\varphi}^{2}=0 \tag{2.5}$$

$$(2) \qquad \ddot{\vartheta}+\Gamma^{2}_{\alpha\beta}\cdot\dot{x}^{\alpha}\cdot\dot{x}^{\beta}=\ddot{\vartheta}+2\cdot\Gamma^{2}_{12}\cdot\dot{r}\cdot\dot{\vartheta}+\Gamma^{2}_{33}\cdot\dot{\varphi}^{2}=\ddot{\vartheta}-\frac{2}{r}\cdot\dot{r}\cdot\dot{\vartheta}-\cos(\vartheta)\cdot\sin(\vartheta)\cdot\dot{\varphi}^{2}=0 \tag{2.6}$$

$$(3) \qquad \ddot{\varphi}+\Gamma^{3}_{\alpha\beta}\cdot\dot{x}^{\alpha}\cdot\dot{x}^{\beta}=\ddot{\varphi}+2\cdot\Gamma^{3}_{13}\cdot\dot{r}\cdot\dot{\varphi}+2\cdot\Gamma^{3}_{23}\cdot\dot{\vartheta}\cdot\dot{\varphi}=\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\cdot\dot{\varphi}+2\cdot\cot(\vartheta)\cdot\dot{\vartheta}\cdot\dot{\varphi}=0 \tag{2.7}$$

3. The Schwarzschild metric

Only about a month after the publication of Einstein's theory of general relativity, Karl Schwarzschild found the first important exact solution of Einstein's equations in 1915. We examine the gravitational impact of a spherically symmetric body with no electric charge. We assume that if we allow only radial changes in the mass distribution of the central source, the surrounding spacetime will inherit this symmetry. This famous solution of Einstein's equations is very idealistic, but there are two important reasons to treat it as a sufficient approximation. First of all because in practice it is already far more accurate than Newtonian celestial mechanics, and that it has been proven by experiments investigating the Solar System, on the other hand, for a possible instructional study, it shows in the right pedagogic direction.

The Einstein-equation in vacuum can be further reduced:

$$R_{\eta\kappa}-\frac{1}{2}\cdot g_{\eta\kappa}\cdot R=0 \qquad /\cdot g^{\eta\kappa}$$

$$R=0 \qquad \rightarrow \qquad R_{\eta\kappa}=0 \tag{3.1}$$

We assume that the geometry of spacetime approaches a flat surface at big distances. None of the coordinates change as a function of another, therefore the metric tensor has only diagonal components. Let us expand the spherically symmetric invariant distance with unknown functions of radial and timelike coordinates:

$$ds^2 = A(r\quad t)\cdot c^2\cdot dt^2 - B(r\quad t)\cdot dr^2 - C(r\quad t)\cdot r^2\cdot d\vartheta^2 - D(r\quad t)\cdot r^2\cdot\sin^2(\vartheta)\cdot d\varphi^2 \tag{3.2}$$

It is possible to reduce the number of unknown functions by utilizing symmetric properties of the surface. We have the freedom to set the *z* axis in any direction, therefore:

$$C(r\quad t) = D(r\quad t) \tag{3.3}$$

Because the radial coordinate can be rescaled, one more arbitrary condition can be introduced among the remaining functions, the most notable variants are:

- Schwarzschild-coordinates: $C(r \quad t) = 1$
  authentic scaling of horizontal distances
- Isotropic coordinates: $B(r \quad t) = C(r \quad t)$
  authentic scaling of directions and angles
- Gaussian polar coordinates: $B(r \quad t) = 1$
  authentic scaling of radial distances
- Comoving coordinates: $A(r \quad t) = 1$
  authentic scaling of timelike distances

The invariant distance with Schwarzschild coordinates:

$$ds^2 = A(r \quad t) \cdot c^2 \cdot dt^2 - B(r \quad t) \cdot dr^2 - r^2 \cdot \left(d\vartheta^{\,2} + \sin^2(\vartheta) \cdot d\varphi^{\,2}\right) \qquad (3.4)$$

We proceed with calculating the partial derivatives of the metric tensor, the connection and its partial derivatives. The curvature tensor is non-zero, and by contracting its indices, we get the non-zero components of the Ricci-tensor, and with it, the equations to solve:

$$(00) \qquad R_{00} = -\frac{\ddot{B}}{2 \cdot B} + \frac{\dot{B}}{4 \cdot B} \cdot \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{A''}{2 \cdot B} + \frac{A'}{r \cdot B} - \frac{A'}{4 \cdot B} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

$$(11) \qquad R_{11} = \frac{\ddot{B}}{2 \cdot B} - \frac{\dot{B}}{4 \cdot A} \cdot \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) - \frac{A''}{2 \cdot A} + \frac{B'}{r \cdot B} + \frac{A'}{4 \cdot A} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

$$(22) \qquad R_{22} = 1 - \frac{1}{B} - \frac{r}{2 \cdot B} \cdot \left(\frac{A'}{A} - \frac{B'}{B}\right) = 0$$

$$(33) \qquad R_{33} = \sin^2(\vartheta) \cdot R_{22} = 0$$

$$(01) \qquad R_{01} = R_{10} = \frac{\dot{B}}{r \cdot B} = 0 \qquad (3.5)$$

According to the non-diagonal components, the time derivative of the $B()$ function is zero:

$$\frac{\dot{B}}{r \cdot B} = 0 \qquad / \cdot r \cdot B$$

$$\dot{B} = 0 \qquad (3.6)$$

This simplifies the first two equations significantly:

$$(1) \qquad \frac{A''}{2} + \frac{A'}{r} - \frac{A'}{4} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

$$(2) \qquad -\frac{A''}{2} + \frac{B' \cdot A}{r \cdot B} + \frac{A'}{4} \cdot \left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \qquad (3.7)$$

Adding them together results after simplification in:

$$\frac{A'}{A} + \frac{B'}{B} = 0 \qquad (3.8)$$

Inserting it into equation (1) we get a differential equation that we can solve:

$$\frac{A''}{2}+\frac{A'}{r}-\frac{A'}{4}\cdot(0)=0$$
$$A''+2\cdot\frac{A'}{r}=0 \qquad f(\ )=A'(\ )$$
$$\frac{df}{dr}=-2\cdot\frac{f}{r} \qquad /\cdot\frac{dr}{f}$$
$$\frac{df}{f}=-2\cdot\frac{dr}{r} \qquad /\int$$
$$\ln(f)=-2\cdot\ln(r)+C_1=\ln\left(\frac{C_1}{r^2}\right) \qquad /e^x$$
$$f=A'=\frac{C_1}{r^2} \qquad /\int$$

The unknown functions of the metric are:

$$A=-\frac{C_1}{r}+C_2=C_2-\frac{C_1}{r} \qquad B=\frac{1}{A}=\frac{1}{C_2-\frac{C_1}{r}} \tag{3.9}$$

Function A is the reciprocal of function B. This means that the general spherically symmetric spacetime is not time-dependent (Birkhoff's theorem). Therefore radial pulsating stars and spherically symmetric supernova explosions don't generate gravitational waves, objects moving in their external spacetime cannot detect radial changes in the matter distribution.

We can determine the integration constants using the following conditions:

1. Newton's celestial mechanics is a proven model of the movements of planets, therefore we set criteria that the results of calculations at low velocities, and big orbiting distances approach the Newtonian approximation. This is a physical condition, and we need further investigation when we need to express the results in SI units. We will investigate this in more detail in the next chapter, but for now, we state that it has a length dimension and call it Schwarzschild-radius: $C_1=r_g$
2. At great distances from the source the spacetime metric must approach the flat case, both metric functions must converge to one. This is a mathematical condition, and we determine the second constant now:

$$\lim_{r\to\infty}A=\lim_{r\to\infty}\left(C_2-\frac{r_g}{r}\right)=1 \tag{3.10}$$
$$C_2-\lim_{r\to\infty}\frac{r_g}{r}=C_2-0=1$$
$$A=1-\frac{r_g}{r} \qquad B=\frac{1}{1-\frac{r_g}{r}} \tag{3.11}$$

We insert them back to the original equation. The infinitesimal invariant distance in the Schwarzschild-metric, with Schwarzschild-coordinates:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \cdot \left(d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2\right) \tag{3.12}$$

We can go on to determine other important quantities in this spacetime. The non-zero components of the connection:

$$\begin{aligned}
&\Gamma^0_{01} = \Gamma^0_{10} = \frac{r_g}{2 \cdot r \cdot (r - r_g)} && \\
&\Gamma^1_{00} = \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} && \Gamma^1_{11} = -\frac{r_g}{2 \cdot r \cdot (r - r_g)} \\
&\Gamma^1_{22} = -(r - r_g) && \Gamma^1_{33} = -(r - r_g) \cdot \sin^2(\vartheta) \\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} && \Gamma^2_{33} = -\cos(\vartheta) \cdot \sin(\vartheta) \\
&\Gamma^3_{23} = \Gamma^3_{32} = \cot(\vartheta) &&
\end{aligned} \tag{3.13}$$

The non-zero components of the curvature tensor:

$$\begin{aligned}
&R^0_{101} = -R^0_{110} = \frac{r_g}{r^2 \cdot (r - r_g)} && R^0_{202} = R^1_{212} = -R^0_{220} = -R^1_{221} = -\frac{r_g}{2 \cdot r} \\
&R^0_{303} = R^1_{313} = -R^0_{330} = -R^1_{331} = -\frac{r_g \cdot \sin^2(\vartheta)}{2 \cdot r} && \\
&R^{[1]}_{[001]} = -R^{[1]}_{[010]} = -\frac{r_g \cdot (r_g - r)}{r^4} && R^2_{002} = R^3_{003} = -R^2_{020} = -R^3_{030} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^4} \\
&R^2_{112} = R^3_{113} = -R^2_{121} = -R^3_{131} = \frac{r_g}{2 \cdot r^2 \cdot (r - r_g)} && \\
&R^2_{323} = -R^2_{332} = \frac{r_g \cdot \sin^2(\vartheta)}{r} && R^3_{223} = -R^3_{232} = -\frac{r_g}{r}
\end{aligned} \tag{3.14}$$

It is interesting to note that the totally covariant curvature tensor and Weyl-tensor are equal in empty spacetime, but further investigation of this subject is beyond the scope of this paper.

Now we can write down the geodesic equations:

$$c \cdot \ddot{t} + 2 \cdot \Gamma^0_{01} \cdot \dot{x}^0 \cdot \dot{x}^1 = 0$$

(0) $$\ddot{t} + \frac{r_g}{r \cdot (r - r_g)} \cdot \dot{t} \cdot \dot{r} = 0$$

$$\ddot{r} + \Gamma^1_{00} \cdot \dot{x}^0 \cdot \dot{x}^0 + \Gamma^1_{11} \cdot \dot{x}^1 \cdot \dot{x}^1 + \Gamma^1_{22} \cdot \dot{x}^2 \cdot \dot{x}^2 + \Gamma^1_{33} \cdot \dot{x}^3 \cdot \dot{x}^3 = 0$$

(1) $$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot (c \cdot \dot{t})^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\vartheta}^2 - (r - r_g) \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 = 0$$

$$\ddot{\vartheta} + 2 \cdot \Gamma^2_{12} \cdot \dot{x}^1 \cdot \dot{x}^2 + \Gamma^2_{33} \cdot \dot{x}^3 \cdot \dot{x}^3 = 0$$

(2) $$\ddot{\vartheta} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\vartheta} - \cos(\vartheta) \cdot \sin(\vartheta) \cdot \dot{\varphi}^2 = 0$$

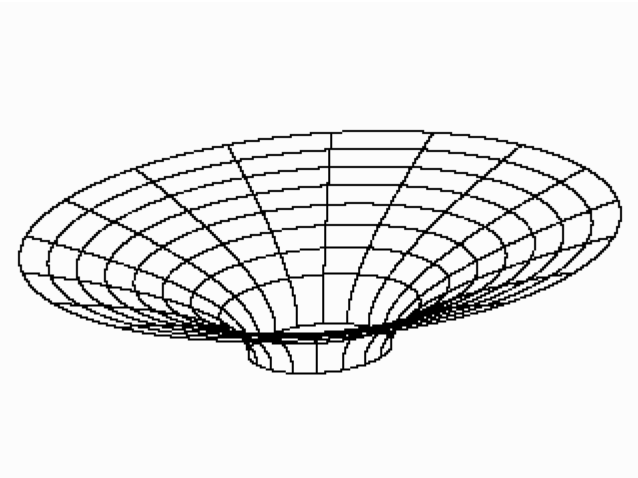
$$\ddot{\varphi} + 2 \cdot \Gamma^3_{13} \cdot \dot{x}^1 \cdot \dot{x}^3 + 2 \cdot \Gamma^3_{23} \cdot \dot{x}^2 \cdot \dot{x}^3 = 0$$

(3) $$\ddot{\varphi} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\varphi} + 2 \cdot \cot(\vartheta) \cdot \dot{\vartheta} \cdot \dot{\varphi} = 0 \qquad (3.15)$$

The second equation can be solved easily:

$$\vartheta = \frac{\pi}{2} \quad \rightarrow \quad d\vartheta = 0 \qquad (3.16)$$

This means, that the path of a revolving body around the Sun will stay in a subspace defined by the above coordinate condition:



Therefore the coordinate system can be rearranged so that the geodesic stays in the equatorial plane, and the geodesic equations can be simplified:

(0) $$\ddot{t} + \frac{r_g}{r \cdot (r - r_g)} \cdot \dot{t} \cdot \dot{r} = 0$$

(1) $$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot (c \cdot \dot{t})^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\varphi}^2 = 0$$

(3) $$\ddot{\varphi} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\varphi} = 0 \qquad (3.17)$$

4. Planetary movement in the Newtonian approximation

The correspondence-principle states that a new model of a system must include the previous theory as a borderline case, and this is not different with general relativity. The investigation of the Kepler orbits of planets as geodesics allows us to identify the remaining integration constant in the derivation of the Schwarzschild-metric. We use the following restrictions on the metric:

1. Movements must be significantly slower than the speed of light: $v << c$
2. Proper time and coordinate time are about the same: $d\tau \approx dt$

The Lagrangian function summarizes the dynamic properties of a system. It is possible to derive from it the equations of movement using the action principle. The action functional in the nonrelativistic case:

$$S[x(t)] = \int_{t_1}^{t_2} L(x \quad \dot{x} \quad t) \cdot dt \tag{4.1}$$

We assume that by varying this integral with a small amount, the result doesn't change significantly, therefore the variation of *S* must be zero. If we calculate this, we get the following equation about the Lagrangian function:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \tag{4.2}$$

In a conservative central force field, the Lagrangian includes the potential and kinetic energies, the latter depending only on the distance from the centre:

$$L = E_k - E_p = \frac{1}{2} \cdot m \cdot v^2 - m \cdot \phi(r) \tag{4.3}$$

The form of the square of velocity can be derived from the invariant distance of the three-dimensional spherical coordinate system:

$$ds^2 = dr^2 + r^2 \cdot d\vartheta^2 + r^2 \cdot \sin^2(\vartheta) \cdot d\varphi^2 \qquad / \cdot \frac{1}{dt^2}$$

$$v^2 = \dot{r}^2 + r^2 \cdot \dot{\vartheta}^2 + r^2 \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 \tag{4.4}$$

Thus the exact form of the Lagrangian:

$$L = m \cdot \left(\frac{1}{2} \cdot \left(\dot{r}^2 + r^2 \cdot \dot{\vartheta}^2 + r^2 \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 \right) - \phi(r) \right) \tag{4.5}$$

Where the Newtonian gravitational potential is:

$$\phi(r) = -\frac{\gamma \cdot M}{r} \tag{4.6}$$

Now we can write down the equations of motion in classical three-dimensional space in spherical coordinates, using results from the second chapter.

In Newtonian approximation time is absolute:

(0) $\ddot{t} = 0$

the radial movement can be determined using the Lagrangian:

(1) $\ddot{r} - r \cdot \dot{\vartheta}^2 - r \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 + \frac{d\phi(r)}{dr} = 0$

There is no force in horizontal directions in a central field:

(2) $\ddot{\vartheta} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\vartheta} - \sin(\vartheta) \cdot \cos(\vartheta) \cdot \dot{\varphi}^2 = 0$

(3) $\ddot{\varphi} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\varphi} + 2 \cdot \cot(\vartheta) \cdot \dot{\vartheta} \cdot \dot{\varphi} = 0$ (4.7)

We seek a spacetime geometry that can produce Kepler orbits. We assume that the extra term in the radial equation of movement is a fictitious force, thus a consequence of the curvature of space. The form of the new term has to be:

$$\Gamma^i_{ab} \cdot \dot{x}^a \cdot \dot{x}^b = \frac{d\phi}{dr} \cdot const \cdot const$$

The first derivatives of any space coordinates of arbitrary movements may be non-zero, therefore this condition is satisfied only if:

$$\dot{x} = const \quad \rightarrow \quad a, b = 0 \quad \rightarrow \quad \frac{c \cdot dt}{dt} = c$$

Thus the new term is a connection as expected for a fictitious force:

$$\frac{d\phi}{dr} = c^2 \cdot \Gamma^1_{00} \qquad (4.8)$$

Let us derive this connection term:

$$\Gamma^1_{00} = \frac{1}{2} \cdot g^{11} \cdot \left(\partial_0 g_{10} + \partial_0 g_{01} - \partial_1 g_{00}\right) \qquad (4.9)$$

According to our experience, the metric in the Solar System is not time dependent, thus the time derivatives of the metric tensor vanish:

$$\partial_0 g_{10} = \partial_0 g_{01} = 0$$
$$\Gamma^1_{00} = -\frac{1}{2} \cdot g^{11} \cdot \partial_1 g_{00} \qquad (4.10)$$

We substitute the known quantities into the equation and perform the integration:

$$\frac{2}{c^2} \cdot \frac{d\phi}{dr} = \frac{dg_{00}}{dr} \qquad / \int$$

$$g_{00} = -\frac{2}{c^2} \cdot \frac{\gamma \cdot M}{r} + C \tag{4.11}$$

We identify the integration constant with the usual condition of vanishing at large distances:

$$\lim_{r \to \infty} g_{00} = \lim_{r \to \infty} \left(-\frac{2}{c^2} \cdot \frac{\gamma \cdot M}{r} + C \right) = C - \lim_{r \to \infty} \frac{2}{c^2} \cdot \frac{\gamma \cdot M}{r} = 1$$

$$C = 1 \tag{4.12}$$

Compare it to the Schwarzschild metric tensor:

$${}_S g_{00} = 1 - \frac{r_g}{r} \qquad {}_N g_{00} = 1 - \frac{2 \cdot \gamma \cdot M}{c^2 \cdot r} \tag{4.13}$$

According to our conditions, they must become equal in the borderline case:

$$1 - \frac{r_g}{r} = 1 - \frac{2 \cdot \gamma \cdot M}{c^2 \cdot r}$$

$$r_g = \frac{2 \cdot \gamma \cdot M}{c^2} \qquad \gamma = 6.67428 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2} \text{ [30]} \tag{4.14}$$

Newton's gravitational constant is one of the hardest to measure among the natural constants, this limits the precision of the masses of celestial bodies. Thus scientists use a product of the two, the so called standard gravitational parameter, which is well known for planetary bodies in the Solar System. For calculations in general relativity it is easier to use the Schwarzschild-radius, a natural measure of mass. The reader can find a table of the most important known values of celestial orbits and bodies in the Appendix.

The invariant distance in Schwarzschild-coordinates, using all SI units:

$$ds^2 = \left(1 - \frac{2 \cdot \gamma \cdot M}{c^2 \cdot r} \right) \cdot c^2 \cdot dt^2 - \frac{dr^2}{1 - \frac{2 \cdot \gamma \cdot M}{c^2 \cdot r}} - r^2 \cdot \left(d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2 \right) \tag{4.15}$$

5. Circular orbits

All forms of energy bend spacetime in a system, thus we will calculate the movement of a test body that is too small to alter the geometry significantly. It revolves in a circular orbit around the centre of the field, far away from the Schwarzschild-radius, along a force-free path, a geodesic. These conditions provide restrictions on the possible coordinates the body can attain:

$$t = t(\tau) \qquad \frac{dt}{d\tau} = const$$
$$r = const \qquad dr = 0$$
$$\vartheta = \frac{\pi}{2} \qquad d\vartheta = 0$$
$$\varphi = \varphi(\tau) \qquad \frac{d\varphi}{d\tau} = const \qquad \omega = \frac{d\varphi}{dt} = const \tag{5.1}$$

Insert them into the geodesic equations, they get significantly reduced:

$$\ddot{t} + \frac{r_g}{r \cdot (r - r_g)} \cdot \dot{t} \cdot \dot{r} = 0$$
(0) $$\ddot{t} = \dot{v}_t = 0$$

$$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot (c \cdot \dot{t})^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\varphi}^2 = 0$$
(1) $$\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot v_t^2 - (r - r_g) \cdot v_\varphi^2 = 0$$

(2) $$\ddot{\vartheta} = \dot{v}_\vartheta = 0$$ (from previous conditions)

$$\ddot{\varphi} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\varphi} = 0$$
(3) $$\ddot{\varphi} = \dot{v}_\varphi = 0 \tag{5.2}$$

The next step is to apply the coordinate conditions on the invariant distance:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 - r^2 \cdot d\varphi^2 \tag{5.3}$$

This is the viewpoint of a distant observer, however the astronaut in the spaceship orbiting on this path doesn't „feel” that he is moving. From his point of view, the invariant distance comes from a locally Minkowskian coordinate system, where he is at complete rest, only his time-like coordinate changes, it is his proper time:

$$ds^2 = c^2 \cdot d\tau^2 - 0 \cdot dx^2 - 0 \cdot dy^2 - 0 \cdot dz^2$$
$$ds^2 = c^2 \cdot d\tau^2 \tag{5.4}$$

The two quantities are equal, because the distance in spacetime doesn't depend on the choice of coordinates, hence the name invariant:

$$c^2 \cdot d\tau^2 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 - r^2 \cdot d\varphi^2$$

We derived the relationship between proper time and coordinate time:

$$d\tau = \sqrt{\left(1 - \frac{r_g}{r}\right) - \frac{r^2 \cdot \omega^2}{c^2}} \cdot dt \tag{5.5}$$

The coordinate time is a monotonic function of the proper time, on the whole geodesic, thus it is suitable for the role of the affine parameter. Use it in the geodesic equation of the radial coordinate (1):

$$\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot v_t^2 - (r - r_g) \cdot v_\varphi^2 = 0$$

$$v_t^2 = \left(\frac{c \cdot dt}{dt}\right)^2 = c^2 \qquad v_\varphi^2 = \left(\frac{d\varphi}{dt}\right)^2 = \omega^2 \tag{5.6}$$

Insert them into the equation:

$$\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 - (r - r_g) \cdot \omega^2 = 0 \qquad / \cdot - \frac{1}{r - r_g}$$

$$- \frac{r_g}{2 \cdot r^3} \cdot c^2 + \omega^2 = 0$$

Orbital frequency on a circular orbit, the orbital period is exactly the same in the Newtonian case:

$$\omega = c \cdot \sqrt{\frac{r_g}{2 \cdot r^3}} \qquad t_c = \frac{2 \cdot \pi}{c} \cdot \sqrt{\frac{2 \cdot r^3}{r_g}} \tag{5.7}$$

The orbit of the Earth is close to a circle, thus it is a good example. We use data from the Appendix, the gravitational radius of the Sun, and the semi-major axis of the Earth's orbit. The orbiting period of Earth:

$$t_c = 31{,}558{,}201 s = 365.2569 days \tag{5.8}$$

The difference from the measured value is only of 10^{-5} magnitude, because we neglected that Earth's orbit is elliptical, and the Sun is rotating, thus the spacetime around it is not exactly Schwarzschild.
We derived Kepler's third law, and identified his constant:

$$\frac{t_2^2}{t_1^2} = \frac{r_2^3}{r_1^3} \qquad \frac{t^2}{r^3} = \frac{8 \cdot \pi^2}{c^2 \cdot r_g} \tag{5.9}$$

Let us insert the formula for orbital frequency into the equation between proper time and coordinate time:

$$d\tau = \sqrt{1 - \frac{3 \cdot r_g}{2 \cdot r}} \cdot dt \qquad (5.10)$$

In the case of light, the change in proper time is zero, thus we derive the radius of the circular orbit of light, the so called light sphere:

$$r = \frac{3 \cdot r_g}{2} \qquad (5.11)$$

Objects slower than the speed of light can orbit only further away than this. The radius of the Sun is much bigger than this, and the planetary orbits are even further away, thus it has no consequence in this paper.

We have to admit, that the *r* coordinate is not the same as the proper distance between the centre of the gravitational field and the orbiting body. That quantity is truly represented only in Gaussian polar coordinates, but that coordinate system can't be examined analytically, the derivations lead to transcendent equations. Therefore the declaration that we recovered Kepler's third law exactly is only true if we explicitly identify *r* with the semi-major axis of the orbiting bodies.

6. Surface acceleration

If the body is not moving in the coordinate system (spacelike coordinates are constant), for example it is sitting still on the surface of a planet, how does it accelerate? We perform our calculations in the function of coordinate time, it doesn't make a big difference for a common planet with a solid surface, like Earth. We apply the following coordinate conditions:

$$\begin{array}{ll} t = t(\tau) & \frac{dt}{d\tau} = const = 1 \\ r = const & dr = 0 \\ \vartheta = \frac{\pi}{2} & d\vartheta = 0 \\ \varphi = const & d\varphi = 0 \end{array} \qquad (6.1)$$

We are looking for the radial acceleration, we insert the coordinate conditions into the geodesic equation:

(1) $$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot (c \cdot \dot{t})^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\varphi}^2 = 0$$

$$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot v_t^2 = 0$$

The radial surface acceleration on a spherical planet:

$$\ddot{r} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 \tag{6.2}$$

If the planet is spinning, the coordinate conditions are expanded and the observer is circling around the spin axis along a latitude:

$$\begin{array}{ll} t = t(\tau) & \frac{dt}{d\tau} = const = 1 \\ r = const & dr = 0 \\ \vartheta = const & d\vartheta = 0 \\ \varphi = changing & d\varphi = const \end{array} \tag{6.3}$$

We use the same geodesic equation, and assume that a slow spinning planet has no significant effects on the spacetime structure. We insert the new coordinate conditions:

(1) $$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot (c \cdot \dot{t})^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\vartheta}^2 - (r - r_g) \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 = 0$$

$$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot v_t^2 - (r - r_g) \cdot \sin^2(\vartheta) \cdot v_\varphi^2 = 0$$

Radial surface acceleration on a slow spinning spherical planet:

$$\ddot{r} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 + (r - r_g) \cdot \sin^2(\vartheta) \cdot \omega^2 \tag{6.4}$$

We insert the equatorial radius of Earth, and the spin frequency into the equation:

$$r = 6.378.137m \qquad \omega = \frac{2 \cdot \pi}{t_k} = 7{,}29211585453 \cdot 10^{-5} \frac{1}{s}$$

Using Earth's gravitational radius from the Appendix, the surface acceleration on the equator is:

$$\ddot{r} = -9{,}76436975 \frac{m}{s^2} \tag{6.5}$$

The difference from the measured value is of 10^{-2} magnitude, due to neglecting the consequences of Earth's ellipsoidal form.

7. Stability of circular orbits

Circular orbits in the spacetime of the Schwarzschild solution are obviously geodesics, but this is only a theoretical case, orbits of real objects are usually more general. The question is when orbits deviate from ideal path, will the geometry of spacetime correct this, thus creating a stable orbit? Otherwise it could increase the disturbance, and drop the revolving body out of the system, or it could plunge into the central gravitating object, increasing its mass.

More specifically,which radii belong to a given orbital frequency? Furthermore, in what direction are the revolving objects perturbed near this point by the geometric potential[5]. Proper time on a timelike geodesic, around the gravitational centre, in the equatorial plane (we go back to denote the metric functions with letters for ease of reading):

$$A \cdot c^2 \cdot dt^2 - B \cdot dr^2 - r^2 \cdot d\varphi^2 = c^2 \cdot d\tau^2 \tag{7.1}$$

We realign the equation to determine the radial velocity, which becomes relevant when the revolving body is getting closer or away from the centre of gravitation. This indicates deviation from the circular orbit:

$$A \cdot \frac{dt^2}{d\tau^2} - \frac{B}{c^2} \cdot \frac{dr^2}{d\tau^2} - \frac{r^2}{c^2} \cdot \frac{d\varphi^2}{d\tau^2} = 1 \tag{7.2}$$

The following tangent vectors determine the geodesics, they are the constants of motion. If the metric doesn't depend on a coordinate, its partial derivative is zero, then the corresponding covariant tangent vector is invariant along the geodesic, thus its partial derivative is also zero (the derivation is easy, and it can be found in most textbooks):

$$u_\varphi = r^2 \cdot \frac{d\varphi}{d\tau} \qquad u_t = A \cdot \frac{dt}{d\tau} \tag{7.3}$$

Inserting them into the equation, and also inserting *B*:

$$\frac{u_t^2}{A} - \frac{B}{c^2} \cdot \frac{dr^2}{d\tau^2} - \frac{u_\varphi^2}{r^2 \cdot c^2} = 1 \qquad B = \frac{1}{A}$$

$$\frac{dr^2}{d\tau^2} = c^2 \cdot u_t^2 - \frac{u_\varphi^2}{r^2 \cdot B} - \frac{c^2}{B}$$

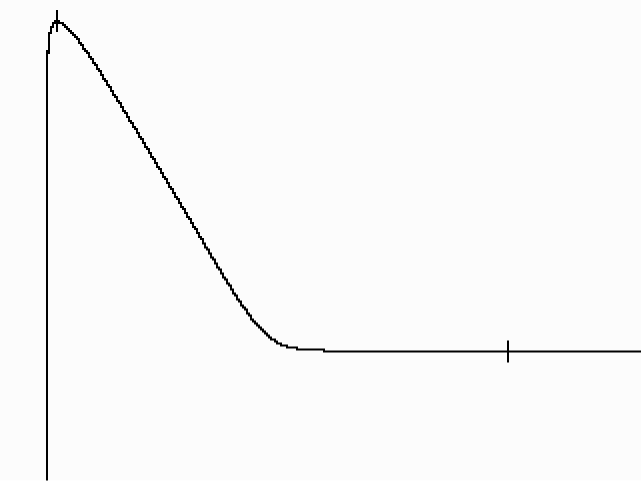
$$\frac{dr^2}{d\tau^2} = c^2 \cdot u_t^2 - \left(1 - \frac{r_g}{r}\right) \cdot \left(\frac{u_\varphi^2}{r^2} + c^2\right) \tag{7.4}$$

This formula strongly resembles the energy conservation equation. Let us call the second term on the right side as the geometric potential:

$$U_g = \left(1 - \frac{r_g}{r}\right) \cdot \left(\frac{u_\varphi^2}{r^2} + c^2\right) \tag{7.5}$$

On a logarithmic scale, with local maximum and minimum noted:

5 The author coined this term for the purposes of this relativistic-only derivation. We will not try to identify the resulting quantities with any classical measure, like the potential energy



This function determines the stability of orbits around the Sun. We can identify them by derivation by the *r* coordinate:

$$U'_g = \frac{r_g \cdot \left(\frac{u_\varphi^2}{r^2} + c^2 \right)}{r^2} - \frac{2 \cdot u_\varphi^2 \cdot \left(1 - \frac{r_g}{r} \right)}{r^3} = 0$$

$$r_g \cdot c^2 \cdot r^2 - 2 \cdot u_\varphi^2 \cdot r + 3 \cdot u_\varphi^2 \cdot r_g = 0 \qquad (7.6)$$

We solve the second-order equation:

$$a \cdot x^2 + b \cdot x + c = 0 \quad \rightarrow \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$r_{1,2} = \frac{2 \cdot u_\varphi^2 \pm \sqrt{4 \cdot u_\varphi^4 - 4 \cdot r_g \cdot c^2 \cdot 3 \cdot u_\varphi^2 \cdot r_g}}{2 \cdot r_g \cdot c^2}$$

The possible radii for circular orbits at a given orbital frequency:

$$r_{1,2} = \frac{u_\varphi^2 \pm u_\varphi \cdot \sqrt{u_\varphi^2 - 3 \cdot r_g^2 \cdot c^2}}{r_g \cdot c^2} \qquad (7.7)$$

The condition for circular orbits is the positive value of the determinant. If it is negative, there is no orbit, and the object is plunging into the gravitating body, or leaves the system permanently. If the determinant is zero:

$$u_\varphi^2 = 3 \cdot r_g^2 \cdot c^2$$

We insert it back to the result of the second-order equation:

$$r_1 = r_2 = 3 \cdot r_g \qquad (7.8)$$

In the general case, r_1 is always larger, r_2 is always smaller than this borderline case. Because the

Sun's radius is much bigger than this, there are only r_1 orbits in the Solar System. We derive the geometric potential again, to identify the stable and unstable orbits. We are looking for places, where the second derivative is zero:

$$U''_g = -\frac{(12\cdot r_g - 6\cdot r)\cdot u_\varphi^2 + 2\cdot c^2\cdot r^2\cdot r_g}{r^5} = 0 \qquad /\cdot\frac{r^5}{2}$$
$$-r_g\cdot c^2\cdot r^2 + 3\cdot u_\varphi^2\cdot r - 6\cdot r_g\cdot u_\varphi^2 = 0 \tag{7.9}$$

Solving the second-order equation gives the distances from the centre of gravitation, where the change of geometric potential turns from negative to positive and vice versa:

$$r_{U1,U2} = \frac{3\cdot u_\varphi^2 \pm u_\varphi\cdot\sqrt{9\cdot u_\varphi^2 - 24\cdot r_g^2\cdot c^2}}{2\cdot r_g\cdot c^2} \tag{7.10}$$

$$r(U''_g < 0) \quad < \quad r_{U2} \quad < \quad r(U''_g > 0) \quad < \quad r_{U1} \quad < \quad r(U''_g < 0)$$

It can be easily shown that in the r_1 case, the second derivative of the geometric potential is always positive, therefore it is an energy minimum, a satellite can have a stable orbit in this region. In the r_2 case, the corresponding value is always negative, thus it is an energy maximum and an unstable orbit, any perturbation will make a particle in this region leave it rapidly:

$$r_2 \quad < \quad r_{U2} \quad < \quad r_1 \quad < \quad r_{U1} \tag{7.11}$$

8. Tides

Galilei back in 1616 called it a superstitious idea, that according to Johannes Kepler, the tides on Earth are caused by the gravitation of the Moon, but the latter scientist was right after all. Even in the Newtonian theory of gravitation, the Moon and the Sun are responsible for tides in Earth's oceans, according to Kepler's idea. Relativity theory with its own tools can describe this phenomenon with even greater accuracy.

On a geodesic, in a local rectangular coordinate system, the form of the geodesic equations is simple, because all connection terms are zero:

$$\frac{d^2x^\eta}{dt^2} = 0 \tag{8.1}$$

However this is valid only for the immediate surrounding of the centre of the coordinate system. Other points of a real finite body move on different, neighbouring geodesics. Observed from the central coordinate system, the connection coefficients will not vanish, acceleration will arise:

$$\frac{d^2x'^\eta}{dt^2} - \Gamma'^\eta_{\alpha\beta}\cdot u^\alpha\cdot u^\beta = 0 \tag{8.2}$$

This value can be approximated with the Taylor-series of the connection in the centre, where it is itself zero, but its higher derivatives are not, due to the presence of curvature:

$$\Gamma'^{\eta}_{\kappa\mu} = \Gamma^{\eta}_{\kappa\mu} + \partial_{\alpha}\Gamma^{\eta}_{\kappa\mu}\cdot x'^{\alpha} + \frac{1}{2}\cdot\partial_{\alpha\beta}\Gamma^{\eta}_{\kappa\mu}\cdot x'^{\alpha}\cdot x'^{\beta} + \ldots \approx \partial_{\alpha}\Gamma^{\eta}_{\kappa\mu}\cdot x'^{\alpha} \tag{8.3}$$

Approximating the geodesic equations to the first order:

$$\frac{d^2x'^{\eta}}{dt^2} - \partial_{\gamma}\Gamma'^{\eta}_{\alpha\beta}\cdot x'^{\gamma}\cdot u^{\alpha}\cdot u^{\beta} = 0 \tag{8.4}$$

Some values present in this formula are given, due to the properties of the coordinate system, we are free to choose others. Using these specific examples has its consequences of course:

$$\begin{aligned} u'^{\eta} = u^{\eta} = \begin{pmatrix} c & 0 & 0 & 0 \end{pmatrix} \quad &\rightarrow \quad u^{\eta}\cdot u^{\kappa} = c^2 \\ x'^{\eta} = \begin{pmatrix} 0 & x^1 & x^2 & x^3 \end{pmatrix} \quad &\rightarrow \quad \partial_0\Gamma^{\eta}_{\kappa\mu} = 0 \end{aligned} \tag{8.5}$$

In case of an orthogonal coordinate system, the x and u vectors are perpendicular to each other, as seen above:

$$\vec{x}\cdot\vec{u} = 0 \tag{8.6}$$

Insert the above restrictions into the equation:

$$\frac{d^2x'^{\eta}}{dt^2} - \partial_{\gamma}\Gamma^{\eta}_{00}\cdot x'^{\gamma}\cdot c^2 = 0 \tag{8.7}$$

Let us expand the derivative of the connection with a similar term, that is non-zero in this first-order approximation. Since we are dealing with infinitesimally small quantities, the following is valid:

$$\begin{aligned} &\partial_{\mu}\Gamma^{\eta}_{00} = \partial_{\mu}\Gamma^{\eta}_{00} - \partial_0\Gamma^{\eta}_{\mu 0} = R^{\eta}_{00\mu} \\ &\frac{d^2x'^{\eta}}{dt^2} - R^{\eta}_{00\gamma}\cdot x'^{\gamma}\cdot c^{\{2\}} = 0 \end{aligned} \tag{8.8}$$

We obtained a coordinate independent quantity, the curvature tensor. Generalize again the tangent vector, we get the equation in its final form:

$$\frac{d^2x^{\eta}}{dt^2} - R^{\eta}_{\alpha\beta\gamma}\cdot x^{\gamma}\cdot u^{\alpha}\cdot u^{\beta} = 0 \tag{8.9}$$

Adopting this result for the Schwarzschild case[6], the question is, what kind of tides arise inside finite bodies around the gravitational centre? We investigate circular orbits only. Let us choose a point on the orbit, $\varphi = 0$ and $t = 0$:

$$u^{\eta} = \begin{pmatrix} c\cdot\frac{dt}{d\tau} & 0 & 0 & \frac{d\varphi}{d\tau} \end{pmatrix}$$

6 The derivation in the Schwarzschild case, and subsequent examinations are independent contributions of the author

$$x^\eta = \begin{pmatrix} 0 & \pm x_r & \pm x_\vartheta & 0 \end{pmatrix} \tag{8.10}$$

Under these conditions, only the following terms of the curvature tensor remain non-zero:

$$R^1_{001} = \frac{r_g \cdot (r - r_g)}{r^4} \qquad R^1_{331} = \frac{r_g \cdot \sin^2(\vartheta)}{2 \cdot r}$$

$$R^2_{002} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^4} \qquad R^2_{332} = -\frac{r_g \cdot \sin^2(\vartheta)}{r}$$

Insert them into the tide equation, let us not forget, that x are coordinates in the local rectangular coordinate system, x_r is the radial, x_ϑ is the perpendicular component:

$$\frac{d^2 x^1}{dt^2} + R^1_{001} \cdot x^1 \cdot u^0 \cdot u^0 - R^1_{331} \cdot x^1 \cdot u^3 \cdot u^3 = 0$$

Coordinate-acceleration in the radial direction:

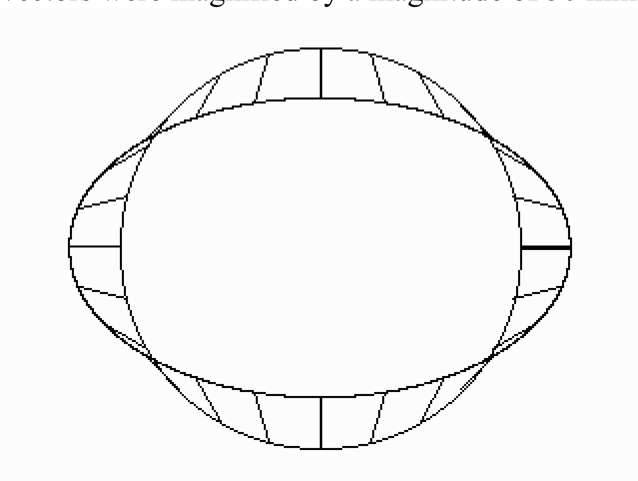
(1) $$\frac{d^2 x_r}{dt^2} = \left(\frac{2 \cdot r_g \cdot (r - r_g)}{r^3 \cdot (2 \cdot r - 3 \cdot r_g)} + \frac{r_g^2}{4 \cdot r^4} \right) \cdot x_r \cdot c^2 \tag{8.11}$$

$$\frac{d^2 x^2}{dt^2} + R^2_{002} \cdot x^2 \cdot u^0 \cdot u^0 - R^2_{332} \cdot x^2 \cdot u^3 \cdot u^3 = 0$$

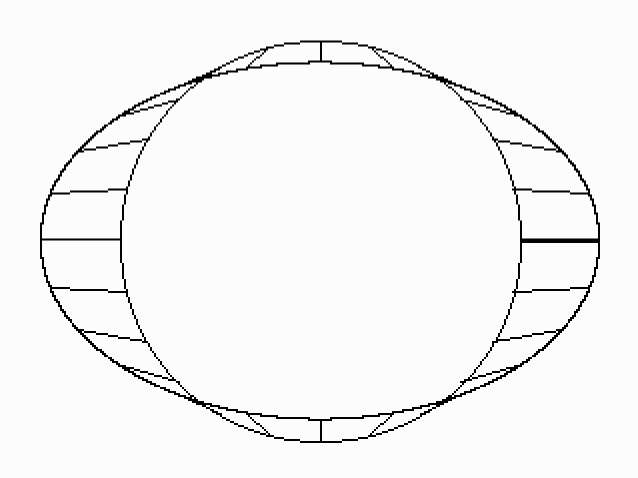
Coordinate-acceleration in the perpendicular direction:

(2) $$\frac{d^2 x_\vartheta}{dt^2} = -\left(\frac{4 \cdot r_g \cdot (r - r_g)}{r^3 \cdot (4 \cdot r - 3 \cdot r_g)} + \frac{r_g^2}{2 \cdot r^4} \right) \cdot x_\vartheta \cdot c^2 \tag{8.12}$$

The two acceleration components above cause deformations in a nearly spherical planet, like Earth. This picture shows the directions, where the points of surface are accelerated (the acceleration vectors were magnified by a magnitude of 50 million):



At distances comparable to the Schwarzschild-radius, the second component becomes dominant. In this case, the form of the ellipse will change, it gets longer in the radial direction, this is called „spaghettisation”. The following picture shows a sphere with a radius of 5 ms, it is 5 kms away from a solar mass black-hole, where timelike orbits are still possible (the acceleration vectors are shrink by one billion times!):



When will the tides become dangerous for the orbiting body? Pieces of matter will separate from the surface, if tidal acceleration becomes bigger than the own surface acceleration. By the comparison it must be taken into account, that the two formulae don't refer to the same coordinate system, therefore we will note the local rectangular coordinates with an *x* at the lower left index. Because the tidal acceleration grows with the radius of the orbiting body, this phenomenon will provide an upper limit for the size of the satellites of gravitational sources:

$$\left(\frac{2 \cdot r_g \cdot (r - r_g)}{r^3 \cdot (2 \cdot r - 3 \cdot r_g)} + \frac{r_g^2}{4 \cdot r^4} \right) \cdot x_r \cdot c^2 - \frac{{}_x r_g \cdot (x_r - {}_x r_g)}{2 \cdot x_r^3} \cdot c^2 = 0 \tag{8.13}$$

This approximation is limited for several reasons, for example the Einstein-equations are non-linear, when we add the two gravitational fields, it has a margin of error. Moreover, we neglected the fact, that the two gravitating bodies deform each other, they are not spherically symmetric any more, neither is their spacetime. The upper limit of integrity is called the Roche-radius, and it can be obtained, by solving the following fourth-order equation:

$$\left(\frac{4 \cdot r_g \cdot (r - r_g)}{r^3 \cdot (2 \cdot r - 3 \cdot r_g)} + \frac{r_g^2}{2 \cdot r^4} \right) \cdot x_r^4 - {}_x r_g \cdot x_r + {}_x r_g^2 = 0 \tag{8.14}$$

We include the constants between the colons into a single symbol:

$$C_r = \left(\frac{4 \cdot r_g \cdot (r - r_g)}{r^3 \cdot (2 \cdot r - 3 \cdot r_g)} + \frac{r_g^2}{2 \cdot r^4} \right)$$

$$x_r^4 - \frac{{}_x r_g}{C_r} \cdot x_r + \frac{{}_x r_g^2}{C_r} = 0 \tag{8.15}$$

According to the solution of the fourth-order equation, we determine the resolvent equation:

$$x^4 + d \cdot x + e = 0$$

$$y^3 + 4\cdot e\cdot y + d^3 = 0$$

$$y_r^3 + 4\cdot \frac{{}_x r_g^2}{C_r}\cdot y_r - \left(\frac{{}_x r_g}{C_r}\right)^3 = 0 \tag{8.16}$$

Solution of the resolvent equation:

$$y^3 + p\cdot y + q = 0$$

$$y_{1,2,3} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$y_r = \sqrt[3]{\frac{1}{2}\cdot\left(\frac{{}_x r_g}{C_r}\right)^3 + \sqrt{\frac{1}{4}\cdot\left(\frac{{}_x r_g}{C_r}\right)^6 + \frac{1}{27}\cdot\left(\frac{{}_x r_g^2}{C_r}\right)^3}} + \sqrt[3]{\frac{1}{2}\cdot\left(\frac{{}_x r_g}{C_r}\right)^3 - \sqrt{\frac{1}{4}\cdot\left(\frac{{}_x r_g}{C_r}\right)^6 + \frac{1}{27}\cdot\left(\frac{{}_x r_g^2}{C_r}\right)^3}} \tag{8.17}$$

Using this result, the solution of the fourth-order equation:

$$x_{1,2,3,4} = \pm\frac{\sqrt{y}}{2} \pm \frac{\sqrt{y \pm \frac{2\cdot d}{\sqrt{y}}}}{2}$$

$$x_{r1,r2,r3,r4} = \pm\frac{\sqrt{y_r}}{2} \pm \frac{\sqrt{y_r \mp \frac{2\cdot {}_x r_g}{C_r\cdot\sqrt{y_r}}}}{2} \tag{8.18}$$

We have two solutions for the Roche-limit of the Moon, orbiting Earth, using the semi-major axis of the Moon's orbit:

$$x_{r1} = 1.09102\cdot 10^{-4}\,m \qquad x_{r2} = 7.07316\cdot 10^{7}\,m \tag{8.19}$$

We have a physical interpretation only for the second value. At this distance from the centre of Moon in the direction of Earth, our planet's gravitation is attracting orbiting bodies more than the Moon, so this is the highest possible stable orbit around it.
The next question is, how close can the Moon come to Earth, what is the distance between the two bodies, where the Roche-radius of the Moon is still more than its physical radius? What is the safe distance from the gravitational centre, where the orbiting body is not yet devoured by it?

$$\left(\frac{2\cdot r_g\cdot(r - r_g)}{r^3\cdot(2\cdot r - 3\cdot r_g)} + \frac{r_g^2}{4\cdot r^4}\right)\cdot x_r\cdot c^2 - \frac{{}_x r_g\cdot(x_r - {}_x r_g)}{2\cdot x_r^3}\cdot c^2 = 0$$

We get a fifth-order equation for *r*:

$$\frac{{}_x r_g\cdot(x_r - {}_x r_g)}{x_r^3}\cdot r^5 - \frac{3\cdot {}_x r_g\cdot(x_r - {}_x r_g)}{2\cdot x_r^3}\cdot r_g\cdot r^4 - 2\cdot r_g\cdot x_r\cdot r^2 + \frac{3\cdot r_g^2}{2}\cdot x_r\cdot r + \frac{3\cdot r_g^3}{4}\cdot x_r = 0$$

(8.20)

Equations above fourth-order can't be solved analytically, the numerical solution is:

$$r = 78{,}875{,}000m \tag{8.21}$$

This is the smallest distance for the Moon around the Earth, without being destroyed.

9. Perihelion precession

After Urbain Le Verrier discovered Neptune by examining the orbit of Uranus, he turned his attention to the orbital anomalies of Mercury. After accounting for the influence of every planet in the Solar System, there was still a small but measurable difference between the Newtonian calculated and measured value of perihelion precession. Contemporary explanations failed, until 1915, when Einstein easily explained the phenomenon using general relativity, this is one of its classical proofs. There was no known exact solution of the Einstein-equations by this time (except Minkowskian spacetime of course), so Einstein used a slightly different method than the one presented here.

We examine orbits that are slightly different from circles. It turned out when we examined orbit stability, that the radial distance is oscillating with the period: T_r, while the revolving is around the centre with a period of T_φ. The resulting orbit can be closely approximated with a rotating ellipse, with the change of angle:

$$\Delta\varphi = \omega_{et} \cdot \left(T_r - T_\varphi\right) = 2 \cdot \pi \cdot \frac{\omega - \omega_{et}}{\omega} \tag{9.1}$$

The orbits are characterized by the already known equation:

$$\frac{dr^2}{d\tau^2} = c^2 \cdot u_t^2 - U_{eff} \tag{9.2}$$

If we approach the geometric potential with its second derivative around the point of balance, we can replace this with the equation of the harmonic oscillator, where we obtain the circular frequency of the periodic movement:

$$U_{eff}(r) \approx \frac{1}{2} \cdot U''_{eff}(r_+) \cdot (r - r_+)^2$$

$$\frac{dr^2}{d\tau^2} = c^2 \cdot u_t^2 - \frac{1}{2} \cdot U''_{eff} \tag{9.3}$$

The frequency of the rotation of the ellipse in proper time:

$$\omega_e = \sqrt{\frac{U''_{eff}}{2}} \tag{9.4}$$

We can identify this quantity by inserting the second derivative of the geometric potential:

$$U''_{eff} = -\frac{(12\cdot r_g - 6\cdot r)\cdot u_\varphi^2 + 2\cdot c^2\cdot r^2\cdot r_g}{r^5}$$

$$\omega_e^2 = -\frac{(6\cdot r_g - 3\cdot r)\cdot u_\varphi^2}{r^5} - \frac{c^2\cdot r_g}{r^3} \tag{9.5}$$

Inserting the orbiting frequency and a constant of movement:

$$\omega = c\cdot\sqrt{\frac{r_g}{2\cdot r^3}} \qquad u_\varphi = r^2\cdot\dot{\varphi} = r^2\cdot\omega$$

$$\omega_e^2 = \left(1 - \frac{6\cdot r_g}{r}\right)\cdot\omega^2 \tag{9.6}$$

Orbits in the Solar System are so far away from the Sun, that the ratio of the gravitational radius and the semi-major axis is very small. Thus we can use an approximation, to get a form that is easier to solve:

$$\omega_e^2 \approx \left(1 - \frac{3\cdot r_g}{2\cdot r_+}\right)\cdot\omega^2 \tag{9.7}$$

The reason for this choice is the connection between the orbital frequency in proper time and in coordinate time for circular orbits. This is an approximation again, because the orbit is only nearly circular:

$$\omega_{et}^2 \approx \left(1 - \frac{3\cdot r_g}{2\cdot r_+}\right)\cdot\omega_e^2 = \left(1 - \frac{3\cdot r_g}{2\cdot r_+}\right)^2\cdot\omega^2$$

$$\omega_{et} = \left(1 - \frac{3\cdot r_g}{2\cdot r_+}\right)\cdot\omega \tag{9.8}$$

We insert this into the initial condition, the precession of the perihelion:

$$\Delta\varphi = 2\cdot\pi\cdot\frac{\omega - \omega_{et}}{\omega} = 3\cdot\pi\cdot\frac{r_g}{r_+} \tag{9.9}$$

One of the famous classical confirmations of general relativity is the perihelion precession of Mercury. Using the semi-major axis, and the orbital period, the change in the position of the perihelion in one turn around the Sun, and in a century:

$$\Delta\varphi = 4.8064538413\cdot 10^{-7}$$

$$\Delta\varphi = 1.9956522\cdot 10^{-4} = 41.163282'' \tag{9.10}$$

10. Deflection of light

Assuming the Newtonian particle representation of light, Johann Georg von Soldner suggested already in 1801, that the rays of light are deflected in a gravitational field, and by treating them like simple bodies in orbit, he calculated their path near the Sun. His result for the bending of light was only half of the actual value. Einstein used relativity theory to obtain first a false, later a correct value that was confirmed in 1919 by the British expedition of Sir Arthur Eddington in Brazil and Equatorial Guinea, where they observed the positions of known stars during a solar eclipse. Later in the 1960's, the results were confirmed by a precision of better than 1 in 10000 by radio-astronomical measurements.

The method presented here[35] differs from the traditional derivation, in essence, we will search the form of a geodesic in a subspace. In our case, in the three-dimensional surface determined by the coordinate condition $\vartheta = \frac{\pi}{2}$, the „shadow" of the geodesics is still a geodesic, as we saw it when we previously derived the general geodesic equations, this explains the success of the followings.

We examine the path of light rays in the general case. Outside of the light sphere, they will not be closed curves, but they will pass by and avoid the singularity, or cross the event horizon. Because of the size of the Sun, only paths that pass by are interesting in the Solar System. As the metric is spherically symmetric, we don't loose generality, if we restrict ourselves to the equatorial plane. The Schwarzschild invariant distance on light-like geodesics is zero:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \cdot d\varphi^2 = 0 \qquad (10.1)$$

It is possible to rearrange this, the radial and original vertical angle coordinate span a surface, where the coordinate time plays the role of the invariant distance:

$$dt^2 = \left(\frac{r}{r - r_g}\right)^2 \cdot dr^2 + \frac{r^3}{r - r_g} \cdot d\varphi^2 \qquad (10.2)$$

This surface is a projection of the original spacetime, which conserves the mutual relationships of the coordinates. The pictures or shadows of the light rays here are geodesics, that can be parametrized with the invariant distance, valid on this surface. We write down the metric tensor, its partial derivatives, and calculate the connection coefficients, to write down the geodesic equations:

$$\ddot{x}^i + \Gamma^i_{ab} \cdot \dot{x}^a \cdot \dot{x}^b = 0$$

$$\ddot{r} + \Gamma^1_{11} \cdot \dot{r}^2 + \Gamma^1_{22} \cdot \dot{\varphi}^2 = 0 \qquad \rightarrow \qquad \ddot{r} - \frac{r_g}{r \cdot (r - r_g)} \cdot \dot{r}^2 - \frac{2 \cdot r - 3 \cdot r_g}{2} \cdot \dot{\varphi}^2 = 0$$

(1) $$\frac{d^2 r}{dt^2} = \frac{r_g}{r \cdot (r - r_g)} \cdot \left(\frac{dr}{dt}\right)^2 + \frac{2 \cdot r - 3 \cdot r_g}{2} \cdot \left(\frac{d\varphi}{dt}\right)^2 \qquad (10.3)$$

$$\ddot{\varphi} + 2 \cdot \Gamma^2_{12} \cdot \dot{r} \cdot \dot{\varphi} = 0 \qquad \rightarrow \qquad \ddot{\varphi} + 2 \cdot \frac{2 \cdot r - 3 \cdot r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r} \cdot \dot{\varphi} = 0$$

(2) $$\frac{d^2\varphi}{dt^2} = -\frac{2 \cdot r - 3 \cdot r_g}{r \cdot (r - r_g)} \cdot \frac{dr}{dt} \cdot \frac{d\varphi}{dt} \tag{10.4}$$

Changes of coordinates – velocities as the functions of distance, examining (2):

$$v_\varphi = \frac{d\varphi}{dt}$$

$$\frac{dv_\varphi}{dt} = -\frac{2 \cdot r - 3 \cdot r_g}{r \cdot (r - r_g)} \cdot \frac{dr}{dt} \cdot v_\varphi \qquad / \cdot \frac{dt}{v_\varphi}$$

$$\frac{dv_\varphi}{v_\varphi} = -\frac{2 \cdot r - 3 \cdot r_g}{r \cdot (r - r_g)} \cdot dr \qquad / \int$$

$$\log(v_\varphi) = \log(r - r_g) - 3 \cdot \log(r) + C \qquad / e^x$$

$$v_\varphi = \frac{d\varphi}{dt} = C \cdot \frac{r - r_g}{r^3} \tag{10.5}$$

It is possible to determine the constant of integration, if we rearrange the invariant distance of the surface, and determine angular velocity in an extreme case:

$$dt^2 = \left(\frac{r}{r - r_g} \right)^2 \cdot dr^2 + \frac{r^3}{r - r_g} \cdot d\varphi^2 \qquad / \cdot \frac{1}{dt^2}$$

$$1 = \left(\frac{r}{r - r_g} \right)^2 \cdot \frac{dr^2}{dt^2} + \frac{r^3}{r - r_g} \cdot \frac{d\varphi^2}{dt^2} \tag{10.6}$$

On the path avoiding the singularity at its closest point (the perihelion) to the centre, the change of the radial distance is zero:

$$\frac{dr_0}{dt} = 0$$

$$\sqrt{\frac{r_0 - r_g}{r_0^3}} = \frac{d\varphi}{dt} \tag{10.7}$$

By inserting this into the integral, we can determine the constant:

$$\frac{d\varphi}{dt} = C \cdot \frac{r_0 - r_g}{r_0^3} \tag{10.8}$$

$$\sqrt{\frac{r_0 - r_g}{r_0^3}} = C \cdot \frac{r_0 - r_g}{r_0^3}$$

$$C = \sqrt{\frac{r_0^3}{r_0 - r_g}} \tag{10.9}$$

The angular velocity:

$$\frac{d\varphi}{dt} = v_\varphi = \sqrt{\frac{r_0^3}{r_0 - r_g} \cdot \frac{r - r_g}{r^3}} \tag{10.10}$$

The radial velocity can be also determined, if we insert the above equation into the formula of the invariant distance:

$$1 = \left(\frac{r}{r - r_g}\right)^2 \cdot \frac{dr^2}{dt^2} + \frac{r^3}{r - r_g} \cdot \frac{d\varphi^2}{dt^2}$$

$$\left(\frac{r}{r - r_g}\right)^2 \cdot \frac{dr^2}{dt^2} = 1 - \frac{r^3}{r - r_g} \cdot \frac{r_0^3}{r_0 - r_g} \cdot \left(\frac{r - r_g}{r^3}\right)^2$$

$$\frac{dr}{dt} = v_r = \frac{r - r_g}{r} \cdot \sqrt{1 - \left(\frac{r_0}{r}\right)^3 \cdot \frac{r - r_g}{r_0 - r_g}} \tag{10.11}$$

The ratio of the two velocities determines the change of the angle-coordinate as a function of distance. By integrating the equation we can determine the angle of the complete turn around the centre, from the infinite distance to the closest point of the path:

$$\frac{d\varphi}{dr} = \frac{1}{r^2} \cdot \sqrt{\frac{\frac{r_0^3}{r_0 - r_g}}{1 - \left(\frac{r_0}{r}\right)^3 \cdot \frac{r - r_g}{r_0 - r_g}}} \quad \rightarrow \quad \varphi = \int_{r_0}^{r=\infty} \frac{1}{r^2} \cdot \sqrt{\frac{\frac{r_0^3}{r_0 - r_g}}{1 - \left(\frac{r_0}{r}\right)^3 \cdot \frac{r - r_g}{r_0 - r_g}}} \cdot dr \tag{10.12}$$

We get an equation, that is easier to handle, if we switch the variable of integration, assuming values between zero and one:

$$\rho = \frac{r_0}{r}$$

$$\varphi = \int_{\rho=0}^{1} \frac{1}{\sqrt{1 - \rho^2}} \cdot \frac{1}{\sqrt{1 - \frac{r_g}{r_0} \cdot \frac{1 - \rho^3}{1 - \rho^2}}} \cdot d\rho \tag{10.13}$$

This integral cannot be solved analytically, but we can write it down as a series of integrands, each of them integrable separately:

$$\varphi = \int_{\rho=0}^{1} \frac{1}{\sqrt{1 - \rho^2}} \cdot \left(1 + \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \frac{1 - \rho^3}{1 - \rho^2} + \frac{3}{8} \cdot \left(\frac{r_g}{r_0} \cdot \frac{1 - \rho^3}{1 - \rho^2}\right)^2 + \frac{5}{16} \cdot \left(\frac{r_g}{r_0} \cdot \frac{1 - \rho^3}{1 - \rho^2}\right)^3 + \ldots\right) \cdot d\rho \tag{10.14}$$

The first term characterizes the light ray, that moves in flat spacetime:

$$\varphi_1 = \int_{\rho=0}^{1} \frac{1}{\sqrt{1-\rho^2}} \cdot d\rho = \arcsin(\rho)\Big|_0^1 = \frac{\pi}{2} \qquad (10.15)$$

In the presence of gravitation, the deviation from this value is called deflection of light. It was first observed during a solar eclipse, when the positions of known stars were determined close to the dark disk of the Sun. These initial measurements were imprecise, and they didn't go beyond the precision of the second term:

$$\varphi_2 = \int_{\rho=0}^{1} \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2} \cdot d\rho \qquad (10.16)$$

$$\varphi_2 = \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \int_{\rho=0}^{1} \frac{1-\rho^3}{\left(1-\rho^2\right)^{\frac{3}{2}}} \cdot d\rho = \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \left(-\sqrt{\frac{1-\rho}{1+\rho}} - \sqrt{(1-\rho)\cdot(1+\rho)} \right)\Bigg|_0^1$$

The actual deflection is the whole angle of the turn from infinity to infinity, twice of the previous value:

$$\varphi_{t2} = 2 \cdot \varphi_2 = 2 \cdot \frac{r_g}{r_0} \qquad (10.17)$$

Evaluating the other terms, we can write down a more precise formula, proved by radio-astronomical measurements, for the deflection of light in a central gravitational field:

$$\varphi_t = 2 \cdot \frac{r_g}{r_0} + \left(\frac{15}{16} \cdot \pi - 1 \right) \cdot \left(\frac{r_g}{r_0} \right)^2 - \left(\frac{15}{16} \cdot \pi + \frac{61}{12} \right) \cdot \left(\frac{r_g}{r_0} \right)^3 + \ldots \qquad (10.18)$$

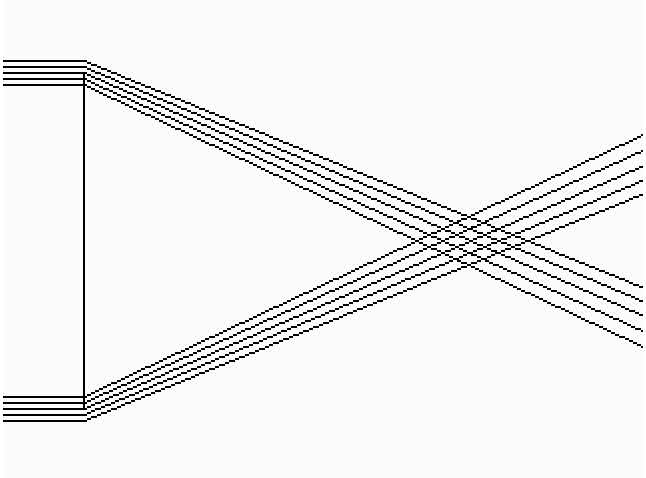
Deflection of light rays at the surface of the Sun, using its gravitational radius:

$$r_0 = 695.500.000m$$

$$\varphi_t = 1{,}75169'' + 7{,}23444 \cdot 10^{-6\,''} - 1{,}26786 \cdot 10^{-10\,''} + \ldots \qquad (10.19)$$

In this case, the higher order terms are not affecting the result significantly, but this not so in a stronger gravitational source, like a neutron star, or a black hole.

Gravitational lens:



The light rays coming from the other side of the Sun – symbolized with a vertical line – are approaching each other on the other side, but they don't meet in a single point (in the focus) like in the case of an optical lens. Despite this, the pictures of objects on the other side of the lens are magnified, and their light is intensified. This phenomenon is used in practice by astronomers. The drawing is schematic, light rays that are incoming from the infinite distance, and touch the surface of the Sun, will intersect again far away on the other side:

$$f = r_0 \cdot \cot(\varphi) = 8{,}18963 \cdot 10^{13}\, m \tag{10.20}$$

In practice, we can call this value the distance of the gravitational focus of the Sun.

11. Gravitational redshift

This phenomenon is one of the classic confirmations of general relativity. Einstein described it already in 1907 based on the equivalence principle, but he didn't think that it is possible to verify it experimentally. Finally the measurements were made in 1959 by R. V. Pound and G. A. Rebka in the United States. The redshift of the gamma ray from the radioactive iron atom directed to 22.5 meters above ground proved Einstein's assumptions by an error margin of 10%. This result was later further corrected below 1% by measurements with hydrogen masers.

On the world line of an object, that is at rest relatively to the global coordinates, only the coordinate time changes. The connection to the proper time can be easily deduced by the invariant distance:

$$c^2 \cdot d\tau^2 = c^2 \cdot g_{00} \cdot dt^2 - 0 \tag{11.1}$$

Proper times of two different observers:

$${}_1d\tau = \sqrt{{}_1g_{00}} \cdot dt \qquad\qquad {}_2d\tau = \sqrt{{}_2g_{00}} \cdot dt \tag{11.2}$$

The *[00]* component of the metric tensor has to be positive, coordinate time must flow in the same direction with proper time, in order to ensure that the causality principle is not violated. By inserting coordinate time, we can show the connection between the two proper times:

$$ {}_1d\tau = \sqrt{\frac{{}_1g_{00}}{{}_2g_{00}}} \cdot {}_2d\tau \tag{11.3} $$

The frequency of light, or any periodic phenomenon:

$$ \nu = \frac{1}{\tau} $$

Thus the gravitational redshift in a general spacetime:

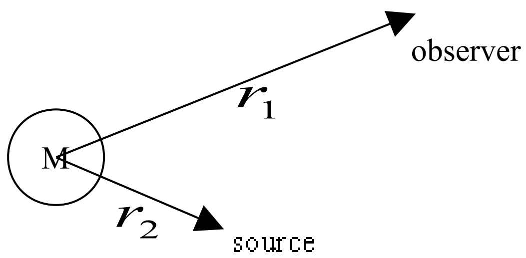
$$ {}_1\nu = \sqrt{\frac{{}_2g_{00}}{{}_1g_{00}}} \cdot {}_2\nu \tag{11.4} $$

Let us insert the metric tensor components of the Schwarzschild-metric:

$$ {}_1\nu = \sqrt{\frac{1 - \frac{r_g}{{}_2r}}{1 - \frac{r_g}{{}_1r}}} \cdot {}_2\nu \tag{11.5} $$

If the source of light is closer to the gravitational centre than the receiver, then:

$$ {}_1r > {}_2r \rightarrow {}_1\nu < {}_2\nu \tag{11.6} $$



The observed frequency in this case is less than the emitted value, electromagnetic radiation in the visible domain is shifted towards red, which is the origin of the name of this phenomenon. Redshift detected by an observer at great distance, where the source of light is r far away from gravitational centre, like the surface of a star:

$$ z = \frac{1}{\sqrt{1 - \frac{r_g}{r}}} - 1 \tag{11.7} $$

12. Geodesic precession

In 1916 when calculating the relativistic correction of the Moon's orbit, Willem de Sitter called for attention on the subject. During the Apollo program, laser mirrors were placed on the surface of the Moon, and by examining the reflected laser light, the phenomenon was verified by an error margin of 7%. NASA launched Gravity Probe B in 2004 with the best gyroscopes ever created by mankind on board, the spheres used in them were perfect by an error margin of 40 atoms. The result of the experiment, that confirmed general relativity by less than 1%, was published on the annual meeting of the American Physical Society in 2007.

One of the most prominent proofs for the curvature of spacetime is parallel displacement, along a geodesic, for example a circular orbit. The vector returning into the initial position will have a different direction:

$$\dot{u}^{\eta} + \Gamma^{\eta}_{\alpha\beta} \cdot v^{\alpha} \cdot u^{\beta} = 0$$

The tangent vector: $$v^{\eta} = \frac{dx^{\eta}}{d\lambda} \qquad (12.1)$$

Our choice for the affine parameter is the coordinate time, and the geodesic is the circular orbit. We are in the equatorial plane, our conditions, and restrictions on the tangent vector:

$$\lambda = t \qquad v^{\eta} = \begin{pmatrix} 1 & 0 & 0 & \omega \end{pmatrix}$$

$$\vartheta = \frac{\pi}{2} \qquad u^{0} = 0 \qquad (12.2)$$

Insert the non-zero connection coefficients into the parallel displacement formula:

$$\dot{u}^{0} + \Gamma^{0}_{01} \cdot v^{0} \cdot u^{1} = 0$$

(0) $$\dot{u}^{0} + \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot u^{1} = 0$$

$$\dot{u}^{1} + \Gamma^{1}_{00} \cdot v^{0} \cdot u^{0} + \Gamma^{1}_{33} \cdot v^{3} \cdot u^{3} = 0$$

(1) $$\dot{u}^{1} - (r - r_g) \cdot \omega \cdot u^{3} = 0$$

$$\dot{u}^{2} + \Gamma^{2}_{33} \cdot v^{3} \cdot u^{3} = 0$$

(2) $$\dot{u}^{2} = 0$$

$$\dot{u}^{3} + \Gamma^{3}_{31} \cdot v^{3} \cdot u^{1} + \Gamma^{3}_{32} \cdot v^{3} \cdot u^{2} = 0$$

(3) $$\dot{u}^{3} + \frac{1}{r} \cdot \omega \cdot u^{1} = 0 \qquad (12.3)$$

Derivation (1):

$$\text{(1)} \qquad \dot{u}^1 - (r - r_g) \cdot \omega \cdot u^3 = 0 \qquad / \cdot \frac{1}{dt}$$

$$\ddot{u}^1 - (r - r_g) \cdot \omega \cdot \dot{u}^3 = 0 \qquad (12.4)$$

Rearrange (3) and insert it into the above result:

$$\dot{u}^3 + \frac{1}{r} \cdot \omega \cdot u^1 = 0 \qquad \rightarrow \qquad \dot{u}^3 = -\frac{1}{r} \cdot \omega \cdot u^1$$

$$\ddot{u}^1 - \omega^2 \cdot \frac{r_g - r}{r} \cdot u^1 = 0 \qquad (12.5)$$

This is the differential equation of the harmonic oscillator, where we see the circular frequency of the rotating tangent vector. The difference from the orbital frequency:

$$\Omega = \omega - \omega \cdot \sqrt{\frac{r - r_g}{r}} = \omega \cdot \sqrt{1 - \frac{3 \cdot r_g}{2 \cdot r}}$$

Inserting the formula for the orbital frequency:

$$\Omega = c \cdot \sqrt{\frac{r_g}{2 \cdot r^3} - \frac{3 \cdot r_g^2}{4 \cdot r^4}} \qquad (12.6)$$

The difference of angular velocities during one turn around the gravitational centre is the geodesic precession:

$$\varpi = \omega - \Omega = c \cdot \sqrt{\frac{r_g}{2 \cdot r^3}} \cdot \left(1 - \sqrt{1 - \frac{3 \cdot r_g}{2 \cdot r}}\right) \qquad (12.7)$$

This is a small effect in a weak gravitational field, but it adds up during several revolvings. The Gravity Probe B satellite was in orbit for 50 weeks between 2004 and 2005, at an altitude of 640 km on a polar orbit. The orbiting period and frequency for the comparison:

$$r = 7011000m \qquad t_k = 5850s = 1h37min30s$$

$$\omega = \frac{2 \cdot \pi}{t_k} = 1{,}074 \cdot 10^{-3} \frac{1}{s} \qquad (12.8)$$

We use the altitude of the satellite and the gravitational radius of Earth in our formula. The angular velocity of the geodesic precession:

$$\varpi = \omega - \Omega = 1{,}198 \cdot 10^{-12} \frac{1}{s} \qquad (12.9)$$

Substitute 50 weeks into the formula:

$$\Delta \varphi = \varpi \cdot t_k = 3{,}084 \cdot 10^{-5} = 6{,}361'' \qquad (12.10)$$

The de Sitter effect is the precession of the Lunar orbit in the gravitational field of the Sun. The orbital frequency of the Moon, from the orbital period:

$$\omega = \frac{2 \cdot \pi}{t_k} = 2{,}6617072 \cdot 10^{-6} \frac{1}{s} \tag{12.11}$$

Now we have to insert the semi-major axis of the Earth's orbit, and the gravitational radius of the Sun into the formula. The angular velocity of the geodesic precession of the lunar orbit:

$$\varpi = \omega - \Omega = 2{,}9478389 \cdot 10^{-15} \frac{1}{s} \tag{12.12}$$

Using a value of one year:

$$\Delta \varphi = \varpi \cdot t_k = 9{,}3024714 \cdot 10^{-8} = 0{,}019187725'' \tag{12.13}$$

This is a displacement of 35.7586 meters compared to the Newtonian result for the Moon along its orbit each year.

13. Axially symmetric coordinate system

Celestial bodies influence spacetime not only with their total energy, but also with their rotation. We will examine this situation now, where spacetime has an axial symmetry. There is no known general solution for rotating planetary bodies, but there are solutions with broad validity, called Tomimatsu-Sato spacetimes, also applicable in the case of a planet. Their borderline case is the Kerr metric, describing the spacetime of a rotating black-hole, therefore it is a vacuum solution. It is not however an external solution for any rotating body, those are given by the more general Tomimatsu-Sato solutions, with a complexity beyond current experimental verification. Thus the Kerr metric is a sufficient approximation for our purposes.

For the derivation[10], we assume that the metric doesn't depend on time, and also because of the axial symmetry, it doesn't depend on the horizontal angle coordinate. The other two coordinate components are left arbitrary. Thus the general infinitesimal invariant distance is:

Coordinates: $\begin{pmatrix} t & x & y & \varphi \end{pmatrix}$

$$ds^2 = e^{2 \cdot \nu} \cdot dt^2 - e^{2 \cdot \psi} \cdot \left(d\varphi - \omega \cdot dt\right)^2 - e^{2 \cdot \mu} \cdot dx^2 - e^{2 \cdot \eta} \cdot dy^2 \tag{13.1}$$

Where we have multiplied the coordinates with four unknown metric functions, and incorporated the rotational frequency. This gives the following totally covariant metric tensor:

$$g_{\eta\kappa} = \begin{pmatrix} e^{2\cdot\nu} - e^{2\cdot\psi}\cdot\omega & 0 & 0 & e^{2\cdot\psi}\cdot\omega \\ 0 & -e^{2\cdot\mu} & 0 & 0 \\ 0 & 0 & -e^{2\cdot\eta} & 0 \\ e^{2\cdot\psi}\cdot\omega & 0 & 0 & -e^{2\cdot\psi} \end{pmatrix} \tag{13.2}$$

It has non-diagonal components, so we have to calculate the determinant of the submatrix containing them:

$$_2g_{\eta\kappa} = \begin{pmatrix} e^{2\cdot\nu} - e^{2\cdot\psi}\cdot\omega & e^{2\cdot\psi}\cdot\omega \\ e^{2\cdot\psi}\cdot\omega & e^{2\cdot\psi} \end{pmatrix} \qquad \eta,\kappa \in \{0 \quad 3\}$$

$$_2g = g_{00}\cdot g_{33} - g_{03}\cdot g_{30} = -\omega^2\cdot e^{4\psi} - e^{2\cdot\psi}\cdot\left(e^{2\cdot\nu} - \omega^2\cdot e^{2\cdot\psi}\right) \tag{13.3}$$

We use the results to write down the contravariant metric tensor:

$$g^{\eta\kappa} = \begin{pmatrix} \frac{1}{e^{2\cdot\nu}} & 0 & 0 & \frac{\omega}{e^{2\cdot\nu}} \\ 0 & -\frac{1}{e^{2\cdot\mu}} & 0 & 0 \\ 0 & 0 & -\frac{1}{e^{2\cdot\eta}} & 0 \\ \frac{\omega}{e^{2\cdot\nu}} & 0 & 0 & \frac{e^{2\cdot\psi}\cdot\omega - e^{2\cdot\nu}}{e^{2\cdot\nu}\cdot e^{2\cdot\psi}} \end{pmatrix} \tag{13.4}$$

After a long calculation, which can be automated using a mathematical equation solver software like Maxima[7], we arrive at the Ricci-tensor's components:

$$R_{00} = -R_{11} = -2\cdot G_{22} = 2\cdot G_{33} = \frac{1}{2}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(\frac{1}{e^{2\cdot\mu}}\cdot\partial_x\omega^2 + \frac{1}{e^{2\cdot\eta}}\cdot\partial_y\omega^2\right) = 0$$

$$R_{01} = R_{10} = \partial_x\left(e^{3\psi-\nu-\mu+\eta}\cdot\partial_x\omega\right) + \partial_y\left(e^{3\psi-\nu+\mu-\eta}\cdot\partial_y\omega\right) = 0$$

$$R_{23} = R_{32} = \frac{1}{2}\cdot e^{2\psi-2\cdot\nu}\cdot\partial_x\omega\cdot\partial_y\omega = 0 \tag{13.5}$$

We introduce the following notation: $\beta = \nu + \psi$, let us use it to write down the (00) and (11) components of the Ricci-tensor in a more symmetric form:

$$\partial_x\left(e^{\beta+\eta-\mu}\cdot\partial_x\nu\right) + \partial_y\left(e^{\beta+\mu-\eta}\cdot\partial_y\nu\right) = +\frac{1}{2}\cdot e^{3\psi-\nu}\cdot\left(e^{\eta-\mu}\cdot\partial_x\omega^2 + e^{\mu-\eta}\cdot\partial_y\omega^2\right)$$

$$\partial_x\left(e^{\beta+\eta-\mu}\cdot\partial_x\psi\right) + \partial_y\left(e^{\beta+\mu-\eta}\cdot\partial_y\psi\right) = -\frac{1}{2}\cdot e^{3\psi-\nu}\cdot\left(e^{\eta-\mu}\cdot\partial_x\omega^2 + e^{\mu-\eta}\cdot\partial_y\omega^2\right) \tag{13.6}$$

The sum and the difference of the equations above, also for two Einstein-tensor components:

7 *Maxima, a Computer Algebra System* is free software from http://maxima.sourceforge.net/

$$R_{00} + R_{11} = G_{22} + G_{33} = \partial_x\left(e^{\eta-\mu}\cdot\partial_x e^{\beta}\right) + \partial_y\left(e^{\mu-\eta}\cdot\partial_y e^{\beta}\right) = 0$$
$$R_{00} - R_{11} = -e^{3\psi-\nu}\cdot\left(e^{\eta-\mu}\cdot\partial_x\omega^2 + e^{\mu-\eta}\cdot\partial_y\omega^2\right)$$
$$G_{22} - G_{33} = -e^{2\psi-2\nu}\cdot\left(e^{\eta-\mu}\cdot\partial_x\omega^2 - e^{\mu-\eta}\cdot\partial_y\omega^2\right) \qquad (13.7)$$

Because of the gauge freedom, we can introduce a constraint:

$$e^{2\cdot(\eta-\mu)} = \Delta(x \quad y) \qquad (13.8)$$

And rewrite the invariant distance, with the new metric functions:

$$ds^2 = e^{\beta}\cdot\left(\chi\cdot dt^2 - \frac{1}{\chi}\cdot(d\varphi - \omega\cdot dt)^2\right) - \frac{e^{\mu+\eta}}{\sqrt{\Delta}}\cdot\left(dx^2 + \Delta\cdot dy^2\right) \qquad (13.9)$$

Where: $\Delta = e^{2\cdot(\eta-\mu)}$ $\qquad \beta = \nu + \psi$ $\qquad \chi = e^{\nu-\psi}$

Insert them into the equations from the Ricci-tensor:

$$\partial_x\left(e^{3\psi-\nu-\mu+\eta}\cdot\partial_x\left(\chi^2 - \omega^2\right)\right) + \partial_y\left(e^{3\psi-\nu+\mu-\eta}\cdot\partial_y\left(\chi^2 - \omega^2\right)\right) = 0 \qquad (13.10)$$

ω and $\chi^2 - \omega^2$ satisfy the same equation, we can use this fact to obtain other solutions. For example the conjugate metric, this will become important later, at the derivation of the Kerr solution. The transformation rule:

$$t \to i\cdot\varphi \qquad \varphi \to -i\cdot t$$

$$\chi\cdot dt^2 - \frac{1}{\chi}\cdot(d\varphi - \omega\cdot dt)^2 \quad \to \quad \frac{1}{\chi}\cdot dt^2 + \frac{2\cdot\omega}{\chi}\cdot dt\cdot d\varphi - \frac{\chi^2-\omega^2}{\chi}\cdot d\varphi^2$$
$$\tilde{\chi}\cdot dt^2 - \frac{1}{\chi}\cdot(d\varphi - \tilde{\omega}\cdot dt)^2$$

$$\tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2} \qquad \tilde{\chi} = \frac{\chi}{\chi^2 - \omega^2} \qquad (13.11)$$

With the choice of the gauge, we can rewrite the invariant distance:

$$\mu = \eta \qquad \Delta = 1$$

$$ds^2 = e^{\beta}\cdot\left(\chi\cdot dt^2 - \frac{1}{\chi}\cdot(d\varphi - \omega\cdot dt)^2\right) - e^{2\cdot\mu}\cdot\left(dx^2 + dy^2\right) \qquad (13.12)$$

Because of this: $\partial^2_{xy} e^{\beta} = 0$ $\qquad (13.13)$

Coordinate transformation, where we use the exponential, as one of the coordinates:

$$e^{\beta} = \rho \qquad (x \quad y) \to (\rho \quad z)$$

$$\partial_x \rho = \partial_y z \qquad \partial_y \rho = -\partial_x z \tag{13.14}$$

Coordinates: $(t \quad \rho \quad z \quad \varphi)$

By inserting them into the infinitesimal distance formula, we obtain the Papapetrou-metric, where the unknown metric functions depend on ρ and z.

$$ds^2 = \rho \cdot \left(\chi \cdot dt^2 - \frac{1}{\chi} \cdot (d\varphi - \omega \cdot dt)^2 \right) - e^{2\cdot\mu} \cdot \left(d\rho^{\,2} + dz^2 \right) \tag{13.15}$$

14. The Ernst equation

This is how far we can get with the first approach. However it can be further specified without loosing generality, by introducing spherical coordinates, and assuming a light-like surface in the metric. This makes the first important difference between the spacetime of empty space and a rotating black hole:

Coordinates: $(t \quad r \quad \vartheta \quad \varphi)$

$$N(x \quad y) = N(r \quad \vartheta) = 0 \qquad g^{\alpha\beta} \cdot \partial_\alpha N \cdot \partial_\beta N = 0$$

$$e^{2\cdot(\eta-\mu)} \cdot \partial_r N^2 + \partial_\vartheta N^2 = 0 \tag{14.1}$$

By the choice of gauge, the equation of the surface is zero:

$$e^{2\cdot(\eta-\mu)} = \Delta(r) = 0 \tag{14.2}$$

The exponential on the surface is zero, and its form in the general case:

$$e^{\beta} = \sqrt{\Delta} \cdot f(\vartheta) = 0 \tag{14.3}$$

We insert all of them into the sum of the (00) and (11) components of the Ricci-tensor:

$$\frac{1}{2} \cdot e^{2\cdot(\psi-\nu)} \cdot \left(\frac{1}{e^{2\cdot\mu}} \cdot \partial_x \omega^{\,2} + \frac{1}{e^{2\cdot\eta}} \cdot \partial_y \omega^{\,2} \right) = 0$$

$$\partial_r \left(\sqrt{\Delta} \cdot \partial_r \sqrt{\Delta} \right) + \frac{1}{f} \cdot \partial^2_{\vartheta\vartheta} f = 0 \tag{14.4}$$

The solutions of the equation:

$$\partial^2_{rr}\Delta = 2 \qquad f = \sin(\vartheta) \tag{14.5}$$

Solution for Δ:

$$\Delta = r^2 - r_g \cdot r + a^2 \tag{14.6}$$

Where r_g and a are constants of integration, the choice for notation is deliberate of course. The behaviour under coordinate transformations will be used later:

$$r_g \to r_g' = 2 \cdot \frac{\sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}}{p} \qquad a \to a' = q \cdot \frac{\sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}}{p} \qquad \frac{r_g}{2} - a \to \frac{r_g'}{2} - a'$$

p and q are real constants: $\qquad p^2 + q^2 = 1 \qquad$ (14.7)

Returning to the Papapetrou-metric, the solutions of the metric tensor, and the formulas for the coordinates:

$$e^{\eta - \mu} = \sqrt{\Delta} \qquad e^{\beta} = \sqrt{\Delta} \cdot \sin(\vartheta)$$
$$\rho = e^{\beta} = \sqrt{\Delta} \cdot \sin(\vartheta) \qquad z = \left(r - \frac{r_g}{2}\right) \cdot \cos(\vartheta) \tag{14.8}$$

We introduce new coordinates again, and insert them into the Ricci-tensor components, to make them look alike. We will repeat this process often, to show the hidden symmetries in the equations:

$$\sigma = \cos(\vartheta) \tag{14.9}$$

Coordinates: $\qquad (t \quad r \quad \sigma \quad \varphi)$

$$\delta = 1 - \sigma^2 = \sin^2(\vartheta)$$

$$R_{00} - R_{11} = \partial_r\left(\Delta \cdot \partial_r(\psi - \nu)\right) + \partial_\sigma\left(\delta \cdot \partial_\sigma(\psi - \nu)\right) = -e^{2\cdot(\psi - \nu)} \cdot \left(\Delta \cdot \partial_r\omega^2 + \delta \cdot \partial_\sigma\omega^2\right)$$
$$R_{01} = R_{10} = \partial_r\left(\Delta \cdot e^{2\cdot(\psi - \nu)} \cdot \partial_r\omega\right) + \partial_\sigma\left(\delta \cdot e^{2\cdot(\psi - \nu)} \cdot \partial_\sigma\omega\right) = 0 \tag{14.10}$$

The same equations by inserting $\chi = e^{\nu - \psi}$:

$$\partial_r\left(\frac{\Delta}{\chi} \cdot \partial_r\chi\right) + \partial_\sigma\left(\frac{\delta}{\chi} \cdot \partial_\sigma\chi\right) = \frac{\Delta \cdot \partial_r\omega^2 + \delta \cdot \partial_\sigma\omega^2}{\chi^2}$$
$$\partial_r\left(\frac{\Delta}{\chi^2} \cdot \partial_r\omega\right) + \partial_\sigma\left(\frac{\delta}{\chi^2} \cdot \partial_\sigma\omega\right) = 0 \tag{14.11}$$

Rearranging:

$$\chi\cdot\left(\partial_r(\Delta\cdot\partial_r\chi)+\partial_\sigma(\delta\cdot\partial_\sigma\chi)\right)=\Delta\cdot\left(\partial_r\chi^2+\partial_r\omega^2\right)+\delta\cdot\left(\partial_\sigma\chi^2+\partial_\sigma\omega^2\right)$$
$$\chi\cdot\left(\partial_r(\Delta\cdot\partial_r\omega)+\partial_\sigma(\delta\cdot\partial_\sigma\omega)\right)=2\cdot\left(\Delta\cdot\partial_r\chi\cdot\partial_r\omega+\delta\cdot\partial_\sigma\chi\cdot\partial_\sigma\omega\right) \tag{14.12}$$

By introducing new variable, we obtain symmetric equations:

$$X=\chi+\omega \qquad\qquad Y=\chi-\omega$$

$$\frac{1}{2}\cdot(X+Y)\cdot\left(\partial_r(\Delta\cdot\partial_r X)+\partial_\sigma(\delta\cdot\partial_\sigma X)\right)=\Delta\cdot\partial_r X^2+\delta\cdot\partial_\sigma X^2$$
$$\frac{1}{2}\cdot(X+Y)\cdot\left(\partial_r(\Delta\cdot\partial_r Y)+\partial_\sigma(\delta\cdot\partial_\sigma Y)\right)=\Delta\cdot\partial_r Y^2+\delta\cdot\partial_\sigma Y^2 \tag{14.13}$$

The following equations will help us to indentify η and μ:

$$R_{23}=R_{32}=-\frac{\sigma}{\delta}\cdot\partial_r(\mu+\eta)+\frac{r-\frac{r_g}{2}}{\Delta}\cdot\partial_\sigma(\mu+\eta)=\frac{2}{(X+Y)^2}\cdot\left(\partial_r X\cdot\partial_\sigma Y+\partial_\sigma X\cdot\partial_r Y\right)$$
$$G_{22}-G_{33}=2\cdot\left(r-\frac{r_g}{2}\right)\cdot\partial_r(\mu+\eta)+2\cdot\sigma\cdot\partial_\sigma(\mu+\eta)=$$
$$\frac{4}{(X+Y)^2}\cdot\left(\Delta\cdot\partial_r X\cdot\partial_r Y-\delta\cdot\partial_\sigma X\cdot\partial_\sigma Y\right)-3\cdot\frac{\left(r-\frac{r_r}{2}\right)^2-\Delta}{\Delta}+\frac{\sigma^2+\delta}{\delta} \tag{14.14}$$

We introduce new coordinates once again, we make good use of the previous coordinate transformation laws (14.7):

$$\kappa=\frac{r-\frac{r_g}{2}}{\sqrt{\left(\frac{r_g}{2}\right)^2-a^2}} \qquad\qquad \Delta=\left(\left(\frac{r_g}{2}\right)^2-a^2\right)\cdot\left(\kappa^2-1\right) \tag{14.15}$$

Coordinates: $\quad (t \quad \kappa \quad \sigma \quad \varphi)$

Inserting into the symmetric equations above:

$$\frac{1}{2}\cdot(X+Y)\cdot\left(\partial_\kappa\left((\kappa^2-1)\cdot\partial_\kappa X\right)+\partial_\sigma\left((1-\sigma^2)\cdot\partial_\sigma X\right)\right)=(\kappa^2-1)\cdot\partial_\kappa X^2+(1-\sigma^2)\cdot\partial_\sigma X^2$$
$$\frac{1}{2}\cdot(X+Y)\cdot\left(\partial_\kappa\left((\kappa^2-1)\cdot\partial_\kappa Y\right)+\partial_\sigma\left((1-\sigma^2)\cdot\partial_\sigma Y\right)\right)=(\kappa^2-1)\cdot\partial_\kappa Y^2+(1-\sigma^2)\cdot\partial_\sigma Y^2$$
$$\tag{14.16}$$

Where the following substitutions are also solutions, this will be very useful later:

$$X \to \frac{X}{1+c\cdot X} \qquad\qquad Y \to \frac{Y}{1-c\cdot Y}$$

Substituting: $$X = \frac{1+F}{1-F} \qquad\qquad Y = \frac{1+G}{1-G}$$

$$(1-F\cdot G)\cdot\left(\partial_\kappa\left(\left(\kappa^2-1\right)\cdot\partial_\kappa F\right)+\partial_\sigma\left(\left(1-\sigma^2\right)\cdot\partial_\sigma F\right)\right) = -2\cdot G\cdot\left(\left(\kappa^2-1\right)\cdot\partial_\kappa F^2+\left(1-\sigma^2\right)\cdot\partial_\sigma F^2\right)$$
$$(1-F\cdot G)\cdot\left(\partial_\kappa\left(\left(\kappa^2-1\right)\cdot\partial_\kappa G\right)+\partial_\sigma\left(\left(1-\sigma^2\right)\cdot\partial_\sigma G\right)\right) = -2\cdot F\cdot\left(\left(\kappa^2-1\right)\cdot\partial_\kappa G^2+\left(1-\sigma^2\right)\cdot\partial_\sigma G^2\right) \tag{14.17}$$

The solutions, where *p* and *q* are real constants:

$$F = -p\cdot\kappa - q\cdot\sigma \qquad\qquad G = -p\cdot\kappa + q\cdot\sigma \qquad\qquad p^2 - q^2 = 1 \tag{14.18}$$

ω can be derived from a coordinate potential, in the following manner:

$$R_{01} = R_{10} = \partial_r\left(\frac{\Delta}{\chi^2}\cdot\partial_r\omega\right)+\partial_\sigma\left(\frac{\delta}{\chi^2}\cdot\partial_\sigma\omega\right) = 0$$
$$R_{00} - R_{11} = \partial_\kappa\left(\Delta\cdot\partial_\kappa\log(\chi)\right)+\partial_\sigma\left(\delta\cdot\partial_\sigma\log(\chi)\right) = \frac{\chi^2}{\Delta}\cdot\partial_\sigma\Phi^2+\frac{\chi^2}{\delta}\cdot\partial_\kappa\Phi^2 \tag{14.19}$$

$$\partial_\kappa\Phi = \frac{\delta}{\chi^2}\cdot\partial_\sigma\omega \qquad\qquad \partial_\sigma\Phi = -\frac{\Delta}{\chi^2}\cdot\partial_\kappa\omega$$
$$\partial_\kappa\left(\frac{\chi^2}{\delta}\cdot\partial_\kappa\Phi\right)+\partial_\sigma\left(\frac{\chi^2}{\Delta}\cdot\partial_\sigma\Phi\right) = 0 \tag{14.20}$$

We introduce a new potential, to obtain an equation, similar to (14.12):

$$\Psi = \frac{\sqrt{\Delta\cdot\delta}}{\chi}$$

$$\Psi\cdot\left(\partial_\kappa\left(\Delta\cdot\partial_\kappa\Psi\right)+\partial_\sigma\left(\delta\cdot\partial_\sigma\Psi\right)\right) = \Delta\cdot\left(\partial_\kappa\Psi^2+\partial_\kappa\Phi^2\right)+\delta\cdot\left(\partial_\sigma\Psi^2+\partial_\sigma\Phi^2\right)$$
$$\Psi\cdot\left(\partial_\kappa\left(\Delta\cdot\partial_\kappa\Phi\right)+\partial_\sigma\left(\delta\cdot\partial_\sigma\Phi\right)\right) = 2\cdot\left(\Delta\cdot\partial_\kappa\Psi\cdot\partial_\kappa\Phi+\delta\cdot\partial_\sigma\Psi\cdot\partial_\sigma\Phi\right) \tag{14.21}$$

If we view the potentials as components of a single complex quantity, we can combine the equations, and obtain a formula similar to (14.16), where a similar transformation also applies:

$$Z = \Psi + i\cdot\Phi$$

$$\mathrm{Re}(Z)\cdot\left(\partial_\kappa\left(\Delta\cdot\partial_\kappa Z\right)+\partial_\sigma\left(\delta\cdot\partial_\sigma Z\right)\right) = \Delta\cdot\partial_\kappa Z^2+\delta\cdot\partial_\sigma Z^2 \tag{14.22}$$

$$Z \to \frac{Z}{1 + i \cdot c \cdot Z}$$

By further using this analogy, we arrive at the Ernst equations:

$$Z = -\frac{1 + E}{1 - E}$$

$$\left(1 - E \cdot E^{\#}\right) \cdot \left(\partial_{\kappa}\left(\Delta \cdot \partial_{\kappa} E\right) + \partial_{\sigma}\left(\delta \cdot \partial_{\sigma} E\right)\right) = -2 \cdot E^{\#} \cdot \left(\Delta \cdot \partial_{\kappa} E^{2} + \delta \cdot \partial_{\sigma} E^{2}\right) \qquad (14.23)$$

Conjugate potentials using the conjugate metric functions (13.11):

$$\widetilde{\omega} = \frac{\omega}{\chi^{2} - \omega^{2}} \qquad \widetilde{\chi} = \frac{\chi}{\chi^{2} - \omega^{2}}$$

$$\widetilde{\Psi} = \frac{\sqrt{\Delta \cdot \delta}}{\widetilde{\chi}} = e^{\psi + \nu} \cdot \frac{\chi^{2} - \omega^{2}}{\chi} = e^{2 \cdot \nu} - \omega^{2} \cdot e^{2 \cdot \psi}$$

$$\partial_{\kappa} \widetilde{\Phi} = \frac{\delta}{\widetilde{\chi}^{2}} \cdot \partial_{\sigma} \widetilde{\omega} = \frac{\widetilde{\Psi}^{2}}{\Delta} \cdot \partial_{\sigma} \widetilde{\omega} \qquad \partial_{\sigma} \widetilde{\Phi} = -\frac{\Delta}{\widetilde{\chi}^{2}} \cdot \partial_{\kappa} \widetilde{\omega} = -\frac{\widetilde{\Psi}^{2}}{\delta} \cdot \partial_{\kappa} \widetilde{\omega}$$

$$\widetilde{Z} = \widetilde{\Psi} + i \cdot \widetilde{\Phi} = -\frac{1 + \widetilde{E}}{1 - \widetilde{E}} \qquad (14.24)$$

Conjugate Ernst equation:

$$\left(1 - \widetilde{E} \cdot \widetilde{E}^{\#}\right) \cdot \left(\partial_{\kappa}\left(\Delta \cdot \partial_{\kappa} \widetilde{E}\right) + \partial_{\sigma}\left(\delta \cdot \partial_{\sigma} \widetilde{E}\right)\right) = -2 \cdot \widetilde{E}^{\#} \cdot \left(\Delta \cdot \partial_{\kappa} \widetilde{E}^{2} + \delta \cdot \partial_{\sigma} \widetilde{E}^{2}\right) \qquad (12.25)$$

$$\left(1 - \widetilde{E} \cdot \widetilde{E}^{\#}\right) \cdot \left(\partial_{\kappa}\left(\left(\kappa^{2} - 1\right) \cdot \partial_{\kappa} \widetilde{E}\right) + \partial_{\sigma}\left(\left(1 - \sigma^{2}\right) \cdot \partial_{\sigma} \widetilde{E}\right)\right) = -2 \cdot \widetilde{E}^{\#} \cdot \left(\left(\kappa^{2} - 1\right) \cdot \partial_{\kappa} \widetilde{E}^{2} + \left(1 - \sigma^{2}\right) \cdot \partial_{\sigma} \widetilde{E}^{2}\right)$$

Where: $\widetilde{\Psi} = \mathrm{Re}\left(\widetilde{Z}\right) = -\frac{1 - \widetilde{E} \cdot \widetilde{E}^{\#}}{\left|1 - \widetilde{E}\right|^{2}} \qquad \widetilde{\Phi} = \mathrm{Im}\left(\widetilde{Z}\right) = -\frac{i \cdot \left(\widetilde{E} - \widetilde{E}^{\#}\right)}{\left|1 - \widetilde{E}\right|^{2}}$

15. The Kerr metric

By examining every single symmetry in the axially symmetric vacuum spacetime, we arrive at a single analytic formula for the metric, first found by Roy Kerr in 1963. The conjugate Ernst equation is similar to (14.16), therefore it is possible to relate the variables to each other, and obtain the solution directly:

$$F = \widetilde{E} \qquad G = \widetilde{E}^{\#}$$

$$\widetilde{E} = -p \cdot \kappa - i \cdot q \cdot \sigma \qquad p^{2} + q^{2} = 1 \qquad (15.1)$$

The complex potential:

$$\tilde{Z} = \tilde{\Psi} + i\cdot\tilde{\Phi} = -\frac{1-p\cdot\kappa - i\cdot q\cdot\sigma}{1+p\cdot\kappa + i\cdot q\cdot\sigma}$$

$$\tilde{\Psi} = \frac{p^2\cdot\left(\kappa^2-1\right)-q^2\cdot\left(1-\sigma^2\right)}{\left(p\cdot\kappa+1\right)^2+q^2\cdot\sigma^2} \qquad \tilde{\Phi} = \frac{2\cdot q\cdot\sigma}{\left(p\cdot\kappa+1\right)^2+q^2\cdot\sigma^2} \tag{15.2}$$

We return to the r coordinate and insert p and q:

Coordinates: $\left(t \quad r \quad \sigma \quad \varphi\right)$

$$p = 2\cdot\frac{\sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}}{r_g} \qquad q = \frac{2\cdot a}{r_g} \tag{15.3}$$

We write down the potentials with them:

$$\tilde{\Psi} = \frac{1}{\rho^2}\cdot\left(\Delta - a^2\cdot\delta\right) \qquad \tilde{\Phi} = \frac{a\cdot r_g\cdot\sigma}{\rho^2} \tag{15.4}$$

Where: $\rho^2 = r^2 + a^2\cdot\sigma^2 = r^2 + a^2\cdot\cos^2(\vartheta)$

Determining unknown metric functions from the potentials:

$$\partial_r\tilde{\Phi} = -\frac{2\cdot a\cdot r_g\cdot r\cdot\sigma}{\rho^4} = \frac{\tilde{\Psi}^2}{\Delta}\cdot\partial_\sigma\tilde{\varpi} = \frac{\left(\Delta - a^2\cdot\delta\right)^2}{\rho^4\cdot\Delta}\cdot\partial_\sigma\tilde{\varpi}$$

$$\partial_\sigma\tilde{\Phi} = \frac{a\cdot r_g}{\rho^4}\cdot\left(r^2 - a^2\cdot\sigma^2\right) = -\frac{\tilde{\Psi}^2}{\delta}\cdot\partial_r\tilde{\varpi} = \frac{\left(\Delta - a^2\cdot\delta\right)^2}{\rho^4\cdot\delta}\cdot\partial_r\tilde{\varpi}$$

$$\partial_\sigma\tilde{\varpi} = -\frac{2\cdot a\cdot r_g\cdot r\cdot\sigma\cdot\Delta}{\left(\Delta - a^2\cdot\delta\right)^2} \qquad \partial_r\tilde{\varpi} = -\frac{a\cdot r_g\cdot\left(r^2 - a^2\cdot\sigma^2\right)\cdot\delta}{\left(\Delta - a^2\cdot\delta\right)^2} \tag{15.5}$$

Conjugate and conventional angular frequency:

$$\tilde{\varpi} = \frac{\omega}{\chi^2 - \omega^2} = \frac{a\cdot r_g\cdot r\cdot\delta}{\Delta - a^2\cdot\delta}$$

$$\tilde{\Psi} = e^{2\psi}\cdot\left(\chi^2 - \omega^2\right) = e^{2\nu} - \omega^2\cdot e^{2\psi} = \frac{\Delta - a^2\cdot\delta}{\rho^2}$$

$$\omega = \frac{a \cdot r_g \cdot r \cdot \delta}{\Delta - a^2 \cdot \delta} \cdot \left(\chi^2 - \omega^2\right) = \frac{a \cdot r_g \cdot r \cdot \delta}{\rho^2} \cdot e^{-2\cdot\psi} \tag{15.6}$$

Combining the upper two equations, and inserting:

$$e^{2\cdot\beta} = \Delta \cdot \delta$$

$$\frac{\Delta - a^2 \cdot \delta}{\rho^2} \cdot e^{2\cdot\psi} = e^{2\cdot\beta} - \omega^2 \cdot e^{4\cdot\psi} = \frac{\delta \cdot \left(\Delta \cdot \rho^4 - a^2 \cdot r_g^2 \cdot r^2 \cdot \delta\right)}{\rho^4} \tag{15.7}$$

Write down some algebraic identities, and introduce a metric function:

$$\left(\left(r^2 + a^2\right) \mp a \cdot \sqrt{\Delta \cdot \delta}\right) \cdot \left(\sqrt{\Delta} \pm a \cdot \sqrt{\delta}\right) = \rho^2 \cdot \sqrt{\Delta} \pm a \cdot r_g \cdot r \cdot \sqrt{\delta}$$
$$\Sigma^2 \cdot \left(\Delta - a^2 \cdot \delta\right) = \rho^4 \cdot \Delta - a^2 \cdot r_g^2 \cdot r^2 \cdot \delta$$
$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2 \cdot \Delta \cdot \delta \tag{15.8}$$

Finally we can write down the metric functions:

$$e^{2\cdot\psi} = \frac{\delta \cdot \Sigma^2}{\rho^2} \qquad \omega = \frac{a \cdot r_g \cdot r}{\Sigma^2}$$

$$e^{2\cdot\nu} = e^{2\cdot\beta} - e^{2\cdot\psi} = \frac{\rho^2 \cdot \Delta}{\Sigma^2} \qquad \chi = e^{\nu - \psi} = \frac{\rho^2}{\Sigma^2} \cdot \sqrt{\frac{\Delta}{\delta}} \tag{15.9}$$

Also we are able to identify *X, Y* and their derivatives:

$$X = \chi + \omega = \frac{\sqrt{\Delta} + a \cdot \sqrt{\delta}}{\left(\left(r^2 + a^2\right) + a \cdot \sqrt{\Delta \cdot \delta}\right) \cdot \sqrt{\delta}}$$
$$Y = \chi - \omega = \frac{\sqrt{\Delta} - a \cdot \sqrt{\delta}}{\left(\left(r^2 + a^2\right) - a \cdot \sqrt{\Delta \cdot \delta}\right) \cdot \sqrt{\delta}} \tag{15.10}$$

$$\partial_r X = \partial_r Y = \frac{\rho^2 \cdot \left(r - \frac{r_g}{2}\right) - 2 \cdot r \cdot \left(\sqrt{\Delta} + a \cdot \sqrt{\delta}\right) \cdot \sqrt{\Delta}}{\left(\left(r^2 + a^2\right) + a \cdot \sqrt{\Delta \cdot \delta}\right)^2 \cdot \sqrt{\Delta \cdot \delta}}$$
$$\partial_\sigma X = \partial_\sigma Y = \frac{\sigma \cdot \sqrt{\Delta} \cdot \left(\left(r^2 + a^2\right) + a^2 \cdot \delta + 2 \cdot a \cdot \sqrt{\Delta \cdot \delta}\right)}{\left(\left(r^2 + a^2\right) + a \cdot \sqrt{\Delta \cdot \delta}\right)^2 \cdot \sqrt{\delta^3}} \tag{15.11}$$

We use then in the (14.14) equations:

$$-\frac{\sigma}{\delta}\cdot\partial_r(\mu+\eta)+\frac{r-\frac{r_g}{2}}{\Delta}\cdot\partial_\sigma(\mu+\eta)=\frac{\sigma}{\rho^2\cdot\Delta\cdot\delta}\cdot\left(\left(r-\frac{r_g}{2}\right)\cdot\left(\rho^2+2\cdot a^2\cdot\delta\right)-2\cdot r\cdot\Delta\right)$$

$$\left(r-\frac{r_g}{2}\right)\cdot\partial_r(\mu+\eta)+\sigma\cdot\partial_\sigma(\mu+\eta)=2-\frac{\left(r-\frac{r_r}{2}\right)^2}{\Delta}+\frac{r\cdot r_g}{\rho^2} \qquad (15.12)$$

Where the solution is: $e^{\mu+\eta}=\frac{\rho^2}{\sqrt{\Delta}}$

By using our initial conventions, the metric functions are:

$$\Delta=e^{2\cdot(\eta-\mu)} \qquad e^{2\cdot\mu}=\frac{\rho^2}{\Delta} \qquad e^{2\cdot\eta}=\rho^2 \qquad (15.13)$$

We insert them into the initial invariant distance, to obtain the Kerr-solution:

$$ds^2=\rho^2\cdot\frac{\Delta}{\Sigma^2}\cdot dt^2-\frac{\Sigma^2}{\rho^2}\cdot\left(d\varphi-\frac{a\cdot r_g\cdot r}{\Sigma^2}\cdot dt\right)^2\cdot\sin^2(\vartheta)-\frac{\rho^2}{\Delta}\cdot dr^2-\rho^2\cdot d\vartheta^2 \qquad (15.14)$$

Let us write down the spacetime of the rotating black-hole with rotational ellipsoid coordinates:

$$x=\sqrt{r^2+a^2}\cdot\sin(\vartheta)\cdot\cos(\varphi)$$
$$y=\sqrt{r^2+a^2}\cdot\sin(\vartheta)\cdot\sin(\varphi)$$
$$z=r\cdot\cos(\vartheta) \qquad (15.15)$$

The invariant distance of the Kerr-solution in Boyer-Lindquist coordinates:

$$ds^2=\left(1-\frac{r_g\cdot r}{\rho^2}\right)\cdot c^2\cdot dt^2-\frac{\rho^2}{\Delta}\cdot dr^2-\rho^2\cdot d\vartheta^2-$$
$$\left(r^2+a^2+\frac{r_g\cdot r\cdot a^2}{\rho^2}\cdot\sin^2(\vartheta)\right)\cdot\sin^2(\vartheta)\cdot d\varphi^2+2\cdot\frac{r_g\cdot r\cdot a}{\rho^2}\cdot\sin^2(\vartheta)\cdot d\varphi\cdot c\cdot dt \qquad (15.16)$$

Where: $\Delta=r^2-r_g\cdot r+a^2$ $\qquad \rho^2=r^2+a^2\cdot\cos^2(\vartheta)$

If a approaches zero, we recover the Schwarzschild-solution, therefore we conclude, that it is the geometric angular momentum. The covariant and contravariant metric tensors:

$$g_{\eta\kappa} = \begin{pmatrix} 1 - \frac{r_g \cdot r}{\rho^2} & 0 & 0 & \frac{r_g \cdot r \cdot a}{\rho^2} \cdot \sin^2(\vartheta) \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{r_g \cdot r \cdot a}{\rho^2} \cdot \sin^2(\vartheta) & 0 & 0 & -\left(r^2 + a^2 + \frac{r_g \cdot r \cdot a^2}{\rho^2} \cdot \sin^2(\vartheta)\right) \cdot \sin^2(\vartheta) \end{pmatrix}$$

$$g^{\eta\kappa} = \begin{pmatrix} \frac{(r^2 + a^2)^2 - \Delta \cdot a^2 \cdot \sin^2(\vartheta)}{\rho^2 \cdot \Delta} & 0 & 0 & \frac{r_g \cdot r \cdot a}{\rho^2 \cdot \Delta} \\ 0 & -\frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{r_g \cdot r \cdot a}{\rho^2 \cdot \Delta} & 0 & 0 & -\frac{\Delta - a^2 \cdot \sin^2(\vartheta)}{\rho^2 \cdot \Delta} \end{pmatrix} \quad (15.17)$$

This is all we need for now, to describe known consequences of the Kerr solution in the Solar System.

16. Frame dragging

NASA launched two LAGEOS (Laser Geodynamics Satellites) satellites into orbit in 1976 and 1992, brass spheres with a diameter of 60 cm, and a weight of 411 kg, thus they experience almost no influence from Earth's upper atmosphere. By measuring the reflection of laser light from mirrors placed on their surface, their highly regular orbits can be measured with great accuracy. They are used to determine the exact shape of Earth, the speed of tectonic plate movement, and over the long term, relativistic effect are adding up in their orbital parameters. By examining data collected during long-term observations, it was possible to detect frame dragging effects in their movement, up to 20% accuracy.

The existence of non-diagonal metric tensor components has interesting consequences for the connection between velocities and momenta:

Momentum: $v_\eta = \dot{x}_\eta$

Expressing velocities from momenta:

(0) $v^0 = g^{0\alpha} \cdot v_\alpha = g^{00} \cdot v_0 + g^{03} \cdot v_3$

(3) $v^3 = g^{3\alpha} \cdot v_\alpha = g^{30} \cdot v_0 + g^{33} \cdot v_3$ (16.1)

If a test body has a zero horizontal momentum, it can have a non-zero velocity or vice versa (3), and it is possible to have momentum without energy (0). If the body is approaching the black-hole or rotating body from a great distance, close to the gravitational centre it will gain angular velocity:

$$\omega_f = \frac{v^3}{v^0} = \frac{g^{30} \cdot v_0 + g^{33} \cdot 0}{g^{00} \cdot v_0 + g^{03} \cdot 0} \tag{16.2}$$

Angular velocity caused by the frame dragging of a rotating black-hole:

$$\omega_f = \frac{g^{30}}{g^{00}} = \frac{r_g \cdot r \cdot a}{\left(r^2 + a^2\right)^2 - \Delta \cdot a^2 \cdot \sin^2(\vartheta)} \tag{16.3}$$

We approximate spacetime around Earth with the Kerr metric, our instruments are not yet precise enough, to distinguish between the effects on a rotating body or a rotating black-hole. The angular momentum:

$$J = \omega \cdot \Theta_t \tag{16.4}$$

Earth can be approximated with a rigid rotating sphere:

$$\Theta_t = \frac{2}{5} \cdot M \cdot R^2 \qquad a = \frac{2}{5} \cdot \frac{\omega \cdot R^2}{c} \tag{16.5}$$

The altitude of the LAGEOS orbit is 5900 km, we calculate the distance from the centre of the Earth, and also the geometric angular momentum of Earth:

$$a = 3.94919 m \qquad r = 1.2271 \cdot 10^7 m$$

The angular velocity because of the frame dragging of the rotating Earth:

$$\omega_f = 1.8958 \cdot 10^{-23} \frac{1}{s} \tag{16.6}$$

The LAGEOS-1 was in orbit for 32 years in 2008. The accumulated displacement along the orbit in arch seconds and meters:

$$\Delta\varphi = 3.9489 \cdot 10^{-9''} \qquad \Delta s = 2.3492 \cdot 10^{-7} m \tag{16.7}$$

If the measurements become precise enough, this method will distinguish between various types of rotating bodies, thus it will be possible to get information about the interior mass distribution of a planet, only by examining orbits around it.

Conclusion

General relativity went beyond being a theory already a long time ago, mostly because of the empirical evidence, and theoretical insights during its golden age, in the 1960's. It is now the established model of gravitation, and using it as a tool to understand problems of celestial mechanics in the Solar System is therefore justified. Because of the correspondence principle, Newtonian theory is still widely used for calculating planetary and satellite orbits, however it's not only about using a tool, that is practical and we got used to, but also about understanding phenomena in nature. The revelations of general relativity in celestial mechanics were comparable to the recognition that the Earth is round. It makes things look complicated at first, but later everything seems to be simple and interconnected, it is important to show this by examining familiar and unexpected consequences of this physical model. This paper attempted to do just that, but there are other reasons for the choice of this subject.

We saw that the precision of measurements went beyond the predictions of the Schwarzschild-metric, even in our immediate neighbourhood within the Solar System. This was a major motivation for omitting the post-Newtonian formalism entirely, and attempt to describe everything with the tensor equations of Riemannian geometry. Also, instead of referring to Chandrasekhar's insightful derivation of the Kerr solution as most papers do, it seemed to be important to review it in three subsequent chapters.

NASA's Gravity Probe B experiment was originally designed to measure frame dragging effects by a precision of 1%. They failed this goal because of thermal noise in the instruments of the satellite, but there are other experiments under way that could measure it with even greater accuracy by using laser gyroscopes. The science of gravitation is soon entering into an age, where we have to distinguish between the impact of rotating planets and black-holes on their surrounding spacetime. The derivation of the Kerr solution uses important physical considerations, that can't be avoided when we try to understand these phenomena. As research is entering this realm, this analytical description of spacetime is becoming useful in practice, and a good starting point for understanding the Tomimatsu-Sato solutions.

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Appendix

Natural constants[30]:

Speed of light: $c = 2.99792458 \cdot 10^8 \frac{m}{s}$

Electric permittivity: $\varepsilon_0 = \frac{1}{4 \cdot \pi \cdot 10^{-7} \cdot [c^2]} \frac{A^2 \cdot s^4}{kg \cdot m^3}$

Magnetic permeability: $\mu_0 = 4 \cdot \pi \cdot 10^{-7} \frac{kg \cdot m}{A^2 \cdot s^2}$

Gravitational constant: $\gamma = 6.67428 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$

Planck constant: $h = 6.62606896 \cdot 10^{-34} J \cdot s$

Elementar charge: $e = 1.602176487 \cdot 10^{-19} C$

Avogadro constant: $N_A = 6.02214179 \cdot 10^{23} \frac{1}{mol}$

Astronomical measures:

Julian year: $1a = 365.25 day = 3.15576 \cdot 10^7 s$

Astronomical unit: $1AU = 1.49597870691 \cdot 10^{11} m$

Lightyear: $1ly = 9.4607304725808 \cdot 10^{15} m$

Parsec:

$$1 parsec = 2.06264806245 \cdot 10^5 AU = 3.08567758128 \cdot 10^{16} m = 3.26156377695 ly$$

Main bodies of the Solar System:

Object name	*Gravitational radius (m)*	*Semi-major axis (m)*	*Orbital period (s)*	*Rotation period (s)*
Sun[8]	$2.9532500765 \cdot 10^3$	$2.5 \cdot 10^{20}$	$7.49 \cdot 10^{15}$	$2.192833 \cdot 10^6$
Mercury	$4.9028 \cdot 10^{-4}$	$5.79091 \cdot 10^{10}$	$7.60053024 \cdot 10^6$	$5.0670144 \cdot 10^6$
Venus	$7.2291 \cdot 10^{-3}$	$1.0820893 \cdot 10^{11}$	$1.9414139616 \cdot 10^7$	$-2.09967984 \cdot 10^7$
Earth	$8.870056078 \cdot 10^{-3}$	$1.495978875 \cdot 10^{11}$	$3.15581500224 \cdot 10^7$	$8.616409054 \cdot 10^4$
Moon	$1.091020268509284 \cdot 10^{-4}$	$3.84399 \cdot 10^8$	$2.3605846848 \cdot 10^6$	$2.3605846848 \cdot 10^6$
Mars	$9.5305 \cdot 10^{-4}$	$2.279391 \cdot 10^{11}$	$5.93542944 \cdot 10^7$	$8.8642663 \cdot 10^4$
Ceres	$1.404 \cdot 10^{-6}$	$4.14703838 \cdot 10^{11}$	$1.451363616 \cdot 10^8$	$3.2667012 \cdot 10^4$
Jupiter	2.81915558	$7.785472 \cdot 10^{11}$	$3.742478208 \cdot 10^8$	$3.573 \cdot 10^4$
Saturn	0.84408275	$1.43344937 \cdot 10^{12}$	$9.359130528 \cdot 10^8$	$3.837 \cdot 10^4$
Uranus	0.1289327	$2.876679082 \cdot 10^{12}$	$2.661041808 \cdot 10^9$	$-6.2064 \cdot 10^4$
Neptune	0.1521333	$4.503443661 \cdot 10^{12}$	$5.200416 \cdot 10^9$	$5.7996 \cdot 10^4$
Pluto	$1.938 \cdot 10^{-5}$	$5.906376272 \cdot 10^{12}$	$7.8289895952 \cdot 10^9$	$-5.51856 \cdot 10^5$
Makemake	$5.9 \cdot 10^{-6}$	$6.8503 \cdot 10^{12}$	$9.7790112 \cdot 10^9$	?
Eris	$2.4656 \cdot 10^{-5}$	$1.0123 \cdot 10^{13}$	$1.759104 \cdot 10^{10}$	?

8 Revolving around the galactic centre

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